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Rational numbers with purely periodic β-expansion

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Abstract

We study real numbers β with the curious property that the β-expansion of all sufficiently small positive rational numbers is purely periodic. It is known that such real numbers have to be Pisot numbers which are units of the number field they generate. We complete known results due to Akiyama to characterize algebraic numbers of degree 3 that enjoy this property. This extends results previously obtained in the case of degree 2 by Schmidt, Hama and Imahashi. Let $\gamma(\beta)$ denote the supremum of the real numbers c in $(0, 1)$ such that all positive rational numbers less than c have a purely periodic β-expansion. We prove that $\gamma(\beta)$ is irrational for a class of cubic Pisot units that contains the smallest Pisot number η. This result is motivated by the observation of Akiyama and Scheicher that $\gamma(\eta) = 0.666\,666\,666\,086\ldots$ is surprisingly close to $2/3$.

1. *Introduction*

One of the most basic results about decimal expansions is that every rational number has an eventually periodic expansion (A sequence $(a_n)_{n\geqslant 1}$ is eventually periodic if there exists a positive integer p such that $a_{n+p} = a_n$ for every positive integer n large enough), the converse being obviously true. In fact, much more is known for we can easily distinguish rationals with a purely periodic expansion (A sequence $(a_n)_{n\geqslant 1}$ is purely periodic if there exists a positive integer p such that $a_{n+p} = a_n$ for every positive integer n): a rational number p/q in the interval $(0, 1)$, in lowest form, has a purely periodic decimal expansion if and only if q and 10 are relatively prime. Thus, both rationals with a purely periodic expansion and rationals with a non-purely periodic expansion are, in some sense, uniformly spread on the unit interval. These results extend *mutatis mutandis* to any integer base $b \geqslant 2$, as explained in the standard monograph of Hardy and Wright [**15**].

However, if one replaces the integer b by an algebraic number that is not a rational integer, it may happen that the situation would be drastically different. As an illustration of this claim, let us consider the following two examples. First, let φ denote the golden ratio, that is, the positive root of the polynomial $x^2 - x - 1$. Every real number ξ in $(0, 1)$ can be uniquely expanded as

$$\xi = \sum_{n\geqslant 1} \frac{a_n}{\varphi^n},$$

where a_n takes only the values 0 and 1, and with the additional condition that $a_n a_{n+1} = 0$ for every positive integer n. The binary sequence $(a_n)_{n\geqslant 1}$ is termed the φ-expansion of ξ. In 1980, Schmidt [**22**] proved the intriguing result that every rational number in $(0, 1)$ has a purely periodic φ-expansion. Such a regularity is somewhat surprising as one may imagine φ-expansions of rationals more intricate than their decimal expansions. Furthermore, the latter property seems to be quite exceptional. Let us now consider $\theta = 1 + \varphi$, the largest root of the polynomial $x^2 - 3x + 1$. Again, every real number ξ in $(0, 1)$ has a θ-expansion, that is, ξ can

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be uniquely expanded as

$$\xi = \sum_{n \geqslant 1} \frac{a_n}{\theta^n},$$

where a_n takes only the values 0, 1 and 2 (and with some extra conditions that we do not care about here). In contrast to our first example, it was proved by Hama and Imahashi [**14**] that no rational number in $(0, 1)$ has a purely periodic θ-expansion.

Both φ- and θ-expansions mentioned above are typical examples of the so-called *β-expansions* introduced by Rényi [**20**]. Let $\beta > 1$ be a real number. The β-expansion of a real number $\xi \in [0, 1)$ is defined as the sequence $d_\beta(\xi) = (a_n)_{n \geqslant 1}$ over the alphabet $\mathcal{A}_\beta := \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ produced by the β-transformation $T_\beta : x \mapsto \beta x \bmod 1$ with a greedy procedure; that is, such that, for all $n \geqslant 1$, $a_n = \lfloor \beta T_\beta^{n-1}(\xi) \rfloor$. The sequence $d_\beta(\xi)$ replaces in this framework the classical sequences of decimal and binary digits since we have

$$\xi = \sum_{n \geqslant 1} \frac{a_n}{\beta^n}.$$

Set

$$\gamma(\beta) := \sup\{c \in [0, 1) \mid \forall\, 0 \leqslant p/q \leqslant c,\ d_\beta(p/q) \text{ is a purely periodic sequence}\}.$$

This note is concerned with those real numbers β with the property that all sufficiently small rational numbers have a purely periodic β-expansion, that is, such that

$$\gamma(\beta) > 0. \tag{1.1}$$

With this definition, we get that $\gamma(\varphi) = 1$, while $\gamma(\theta) = 0$. As one could expect, Condition (1.1) turns out to be very restrictive. We deduce from the works of Akiyama [**5**] and Schmidt [**22**] (see details in Proposition 2.1) that such real numbers β have to be Pisot units. This means that β is both a Pisot number and a unit of the integer ring of the number field it generates. Recall that a Pisot number is a real algebraic integer which is greater than 1 and which has all Galois conjugates (different from itself) inside the open unit disc.

One relevant property for our study is as follows:

$$\text{(F)}: \quad \text{every } x \in \mathbb{Z}[1/\beta] \cap [0, 1) \text{ has a finite } \beta\text{-expansion.}$$

This property was introduced by Frougny and Solomyak [**13**]. It has been studied for various reasons during the last twenty years. In particular, Akiyama [**2**] proved the following unexpected result.

THEOREM A. *If β is a Pisot unit satisfying* (F), *then* $\gamma(\beta) > 0$.

The fact that (F) plays a crucial role in the study of $\gamma(\beta)$ looks somewhat puzzling but it will become more transparent in what follows.

The results of Hama, Imahashi and Schmidt previously mentioned, about φ- and θ-expansions, are actually more general and, using Proposition 2.1, lead to a complete understanding of $\gamma(\beta)$ when β is a quadratic number.

THEOREM HIS. *Let $\beta > 1$ be a quadratic number. Then, $\gamma(\beta) > 0$ if and only if β is a Pisot unit satisfying* (F). *In that case,* $\gamma(\beta) = 1$.

Furthermore, quadratic Pisot units satisfying (F) have been characterized in a simple way: they correspond to positive roots of polynomials $x^2 - nx - 1$, with n running along the positive integers. These exactly correspond to quadratic Pisot units which have a Galois conjugate that is negative.

First, we establish the converse of Theorem A for algebraic numbers of degree 3. This provides a result similar to Theorem HIS in that case.

THEOREM 1.1. *Let $\beta > 1$ be a cubic number. Then $\gamma(\beta) > 0$ if and only if β is a Pisot unit satisfying* (F).

We recall that Pisot units of degree 3 satisfying (F) have been nicely characterized in [**4**]: they correspond to the largest real roots of polynomials $x^3 - ax^2 - bx - 1$, with a, b integers, $a \geqslant 1$ and $-1 \leqslant b \leqslant a + 1$.

It is tempting to ask whether, in Theorem 1.1, Property (F) would imply that $\gamma(\beta) = 1$ as it is the case for quadratic Pisot units. However, Akiyama [**2**] proved that such a result does not hold. Indeed, he obtained that the smallest Pisot number η, which is the real root of the polynomial $x^3 - x - 1$, satisfies $0 < \gamma(\eta) < 1$. More precisely, it was proved in [**8**] that $\gamma(\eta)$ is abnormally close to the rational number 2/3 since one has

$$\gamma(\eta) = 0.666\,666\,666\,086\ldots .$$

This intriguing phenomenon naturally leads us to ask about the arithmetic nature of $\gamma(\eta)$.

In this direction, we will prove that $\gamma(\eta)$ is irrational as a particular instance of the following result.

THEOREM 1.2. *Let β be a cubic Pisot unit satisfying* (F) *and such that the number field $\mathbb{Q}(\beta)$ is not totally real. Then $\gamma(\beta)$ is irrational. In particular, $0 < \gamma(\beta) < 1$.*

Note that in Theorem 1.2 the condition that β does not generate a totally real number field is equivalent to the fact that the Galois conjugates of β are complex (that is, they belong to $\mathbb{C} \setminus \mathbb{R}$). Throughout the paper, a complex Galois conjugate of an algebraic number β denotes a Galois conjugate that belongs to $\mathbb{C} \setminus \mathbb{R}$.

The proofs of our results rely on some topological properties of the tiles of the so-called Thurston tilings associated with Pisot units. We introduce the notion of spiral points for compact subsets of $\mathbb{C}$ that turns out to be crucial for our study. The fact that $\gamma(\beta)$ vanishes for a cubic Pisot unit that does not satisfy (F) is a consequence of the fact that the origin is a spiral point with respect to the central tile of the underlying tiling. Theorem 1.2 comes from the fact that $\gamma(\beta)$ cannot be a spiral point with respect to this tile, which provides a quite unusual proof of irrationality.

2. *Expansions in a non-integer base*

In this section, we recall some classical results and notation about β-expansions. We first explain why we focus on bases β that are Pisot units.

PROPOSITION 2.1. *Let $\beta > 1$ be a real number that is not a Pisot unit. Then $\gamma(\beta) = 0$.*

Proof. Assume that $\gamma(\beta) > 0$. If the β-expansion of a rational number $r \in (0, 1)$ has period p, the following relation holds:

$$r = \frac{a_1\beta^{p-1} + a_2\beta^{p-2} + \ldots + a_p}{\beta^p - 1}. \tag{2.1}$$

Since $\gamma(\beta) > 0$, it follows that $1/n$ has a purely periodic expansion for every positive integer n large enough. We thus infer from (2.1) that β is an algebraic integer.

A similar argument applies to prove that β is a unit (see [**2**, Proposition 6]).

It was proved in [**22**] that the set of rational numbers with a purely periodic β-expansion is nowhere dense in $(0, 1)$ if β is an algebraic integer that is neither a Pisot nor a Salem number. (An algebraic integer $\beta > 1$ is a Salem number if all its Galois conjugates (different from β) have modulus at most equal to 1 and with at least one Galois conjugate of modulus equal to 1.) Hence β is either a Pisot unit or a Salem number.

Let us assume that β is a Salem number. It is known that minimal polynomials of Salem numbers are reciprocal, which implies that $1/\beta \in (0, 1)$ is a conjugate of β. Using Galois conjugation, relation (2.1) still holds when replacing β by $1/\beta$. In this equation, the right-hand side then becomes non-positive. Therefore 0 is the only number with purely periodic β-expansion and $\gamma(\beta) = 0$, which is a contradiction with our initial assumption. Note that this argument already appeared in [**2**, Proposition 5].

Thus β is a Pisot unit, which concludes the proof. □

In what follows, we consider finite, right infinite, left infinite and bi-infinite words over the alphabet $\mathcal{A}_\beta := \{0, 1, \ldots, \lceil\beta\rceil - 1\}$. When the location of the 0 index is needed in a bi-infinite word, it is denoted by the symbol $^\bullet$, as in $\ldots a_{-1}a_0{}^\bullet a_1a_2\ldots$. A suffix of a right infinite word $a_0a_1\ldots$ is a right infinite word of the form $a_ka_{k+1}\ldots$ for some non-negative integer k. A suffix of a left infinite word $\ldots a_{-1}a_0$ is a finite word of the form $a_ka_{k+1}\ldots a_0$ for some non-positive integer k. A suffix of a bi-infinite word $\ldots a_{-1}a_0{}^\bullet a_1\ldots$ is a right infinite word of the form $a_ka_{k+1}\ldots$ for some integer k. Given a finite word $u = u_0\ldots u_r$, we denote by $u^\omega = u_0\ldots u_ru_0\ldots u_r\ldots$ or ${}^\omega u = \ldots u_0\ldots u_ru_0\ldots u_r$ the right or left infinite periodic word obtained by an infinite concatenation of u, respectively. A left infinite word $\ldots a_{-1}a_0$ is eventually periodic if there exists a positive integer p such that $a_{-n-p} = a_{-n}$ for every positive integer n large enough.

It is well known that the β-expansion of 1 plays a crucial role. Set $d_\beta(1) := (t_i)_{i\geqslant 1}$. When $d_\beta(1)$ is finite with length n, that is, when $t_n \neq 0$ and $t_i = 0$ for every $i > n$, an infinite expansion of 1 is given by $d^*_\beta(1) = (t_1\ldots t_{n-1}(t_n - 1))^\omega$. If $d_\beta(1)$ is infinite, we just set $d^*_\beta(1) = d_\beta(1)$. The knowledge of this improper expansion of 1 allows us to decide whether a given word over $\mathcal{A}_\beta$ is the β-expansion of some real number.

Definition 2.2. A finite, left infinite, right infinite or bi-infinite word over the alphabet $\mathcal{A}_\beta$ is an *admissible* word if all its suffixes are lexicographically smaller than $d^*_\beta(1)$.

A classical result of Parry [**18**] is that a finite or right infinite word $a_1a_2\ldots$ is the β-expansion of a real number in $[0, 1)$ if and only if it is admissible. Admissible conditions are of course easier to check when $d^*_\beta(1)$ is eventually periodic. In the case where β is a Pisot number, $d^*_\beta(1)$ is eventually periodic and every element in $\mathbb{Q}(\beta) \cap [0, 1)$ has eventually periodic β-expansion according to [**11**, **22**]. In contrast, algebraic numbers that do not belong to the number field $\mathbb{Q}(\beta)$ are expected to have a chaotic β-expansion (see [**1**]). Note that if the set of real numbers with an eventually periodic β-expansion forms a field, then β is either a Salem or a Pisot number [**22**].

Warning. In what follows, β will denote a Pisot unit and admissibility will refer to this particular Pisot number. The sequence $d^*_\beta(1)$ is thus eventually periodic and we set $d^*_\beta(1) := t_1t_2\ldots t_m(t_{m+1}\ldots t_{m+n})^\omega$. Note that since β is a Pisot unit, we have $\mathbb{Z}[1/\beta] = \mathbb{Z}[\beta]$.

3. *Thurston's tiling associated with a Pisot unit*

Given a real number β, there is a natural way to tile the real line using the notion of β-integers (see below). In his famous lectures, Thurston [**23**] discussed the construction of a dual tiling, a sort of Galois conjugate of this tiling, when β is a Pisot number. Note that some examples of similar tiles had previously been introduced by Rauzy [**19**] to study arithmetic properties of an irrational translation on a two-dimensional torus. In this section, we recall Thurston's construction.

The vectorial space $\mathbb{R}^n \times \mathbb{C}^m$ is endowed with its natural product topology. In what follows, the closure $\overline{X}$, the interior $\mathring{X}$ and the boundary ∂X of a subset X of $\mathbb{R}^n \times \mathbb{C}^m$ will refer to this topology.

The β-transformation induces a decomposition of every positive real number in a β-fractional and a β-integral part as follows. Let $k \in \mathbb{N}$ be such that $\beta^{-k}x \in [0,1)$ and $d_\beta(\beta^{-k}x) = a_{-k+1}a_{-k+2}\dots$. Then

$$x = \underbrace{a_{-k+1}\beta^{k-1} + \ldots + a_{-1}\beta + a_0}_{\beta\text{-integral part}} + \underbrace{a_1\beta^{-1} + a_2\beta^{-2} + \ldots}_{\beta\text{-fractional part}}.$$

In the following, we use the notation $x = a_{-k+1}\dots a_{-1}a_0{\bullet}a_1a_2\dots$. We also note that $x = a_{-k+1}\dots a_{-1}a_0{\bullet}$ when the β-fractional part vanishes and $x = {\bullet}a_1a_2\dots$ when the β-integral part vanishes.

Since $a_{-k+1} = 0$ if $x < \beta^{k-1}$, this decomposition does not depend on the choice of k. The set of β-integers is the set of positive real numbers with vanishing β-fractional part as follows:

$$\mathrm{Int}(\beta) := \{a_{-k+1}\dots a_{-1}a_0{\bullet} \mid a_{-k+1}\dots a_{-1}a_0 \text{ is admissible}, k \in \mathbb{N}\}.$$

Let $\sigma_2, \dots, \sigma_r$ be the non-identical real embeddings of $\mathbb{Q}(\beta)$ in $\mathbb{C}$ and let $\sigma_{r+1}, \dots, \sigma_{r+s}$ be the complex embeddings of $\mathbb{Q}(\beta)$ in $\mathbb{C}$. We define the map Ξ by

$$\Xi : \mathbb{Q}(\beta) \to \mathbb{R}^{r-1} \times \mathbb{C}^s, \quad x \mapsto (\sigma_2(x), \dots, \sigma_{r+s}(x)).$$

Definition 3.1. Let β be a Pisot number. The compact subset of $\mathbb{R}^{r-1} \times \mathbb{C}^s$ defined by

$$\mathcal{T} := \overline{\Xi(\mathrm{Int}(\beta))}$$

is called the *central tile* associated with β (see Figure 1).

Figure 1. *The central tile for the smallest Pisot number η, which satisfies $\eta^3 = \eta + 1$.*

The previous construction can be naturally extended to associate a similar tile with any $y \in \mathbb{Z}[\beta] \cap [0, 1)$.

DEFINITION 3.2. Given such a number y, we define the tile

$$\mathcal{T}(y) := \Xi(y) + \overline{\{\Xi(a_{-k+1} \dots a_0 \bullet) \mid a_{-k+1} \dots a_0 d_\beta(y) \text{ is an admissible word}\}}.$$

Note that $\mathcal{T}(0) = \mathcal{T}$. It is also worth mentioning the following:
(i) there are exactly $n + m$ different tiles up to translation;
(ii) the tiles $\mathcal{T}(y)$ induce a covering of the space $\mathbb{R}^{r-1} \times \mathbb{C}^s$, that is,

$$\bigcup_{y \in \mathbb{Z}[\beta] \cap [0,1)} \mathcal{T}(y) = \mathbb{R}^{r-1} \times \mathbb{C}^s. \tag{3.1}$$

The first observation follows from the fact that $d^*_\beta(1)$ is eventually periodic (see, for instance, [5]). The second property is a consequence of the fact that $\Xi(\mathbb{Z}[\beta] \cap [0, \infty))$ is dense in $\mathbb{R}^{r-1} \times \mathbb{C}^s$, as proved in [3, 5].

Furthermore, in the case where β is a cubic Pisot unit, Akiyama, Rao and Steiner [7] proved that this covering is actually a tiling, meaning that the interior of tiles never meet and their boundaries have zero Lebesgue measure. We then have the following property (see Figure 2).

THEOREM ARS. *Let β be a cubic Pisot unit. If x and y are two distinct elements in $\mathbb{Z}[\beta] \cap [0, 1)$, then*

$$\mathring{\mathcal{T}}(x) \cap \mathring{\mathcal{T}}(y) = \emptyset.$$

In what follows, we will also need the following observation.

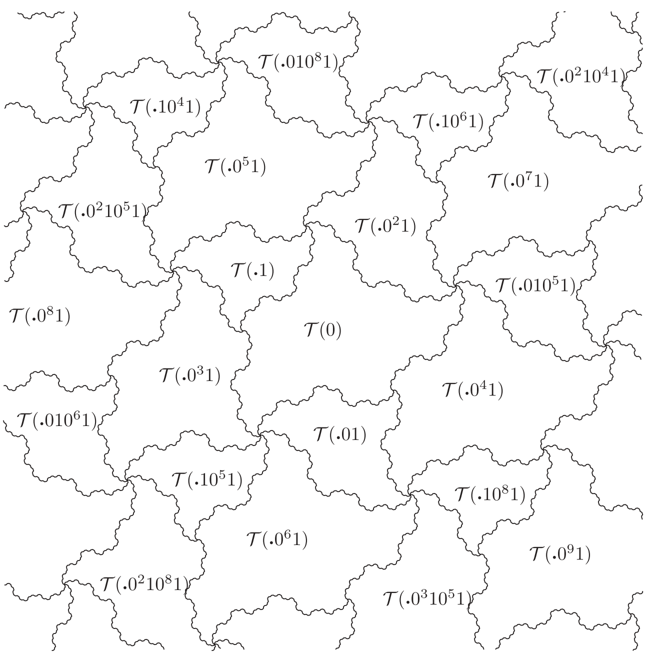

FIGURE 2. *Aperiodic tiling associated with the smallest Pisot number η.*

FACT 3.3. There exists a constant C such that every $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ is contained in at most C different tiles $\mathcal{T}(y)$.

Indeed, since $\mathcal{T}(y) \subseteq \Xi(y) + \mathcal{T}$ and $\mathcal{T}$ is compact, a point $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ cannot belong to the tile $\mathcal{T}(y)$ as soon as the distance between y and z is large enough. The result then follows since the set $\Xi(\mathbb{Z}[\beta] \cap [0,1))$ is uniformly discrete.

4. *A Galois type theorem for expansions in a Pisot unit base*

We introduce now a suitable subdivision of the central tile $\mathcal{T}$. The set $\mathrm{Int}(\beta)$ is a discrete subset of $\mathbb{R}$, so that it can been ordered in a natural way. Then we have the nice property that two consecutive points in $\mathrm{Int}(\beta)$ can differ only by a finite number of values. Namely, if $t_1 \dots t_i$, with $i \geqslant 0$, is the longest prefix of $d^*_\beta(1)$ which is a suffix of $a_{-k+1} \dots a_0$, then this difference is equal to $T^i_\beta(1)$ (see [**6**, **23**]). Since $d^*_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+n})^\omega$, then we have $T^i_\beta(1) = T^{i+n}_\beta(1)$ for $i \geqslant m$, which confirms our claim.

A natural partition of $\mathrm{Int}(\beta)$ is now given by considering the distance between a point and its successor in $\mathrm{Int}(\beta)$.

DEFINITION 4.1. For every $0 \leqslant i < m+n$, we define the subtile $\mathcal{T}_i$ of $\mathcal{T}$ to be the closure of the set of those points $\Xi(a_{-k+1} \dots a_0\bullet)$ such that the distance from $a_{-k+1} \dots a_0\bullet$ to its successor in $\mathrm{Int}(\beta)$ is equal to $T^i_\beta(1)$ (see Figure 3).

As detailed in [**9**, **10**], the Perron–Frobenius theorem coupled with self-affine decompositions of tiles implies that the subtiles have nice topological properties. One of them will be useful in the following proposition: their interiors are disjoint.

PROPOSITION BS. *Let β be a Pisot unit. For every pair (i,j), with $i \neq j$ and $0 \leqslant i,j < m+n$, we have*

$$\mathring{\mathcal{T}}_i \cap \mathring{\mathcal{T}}_j = \emptyset.$$

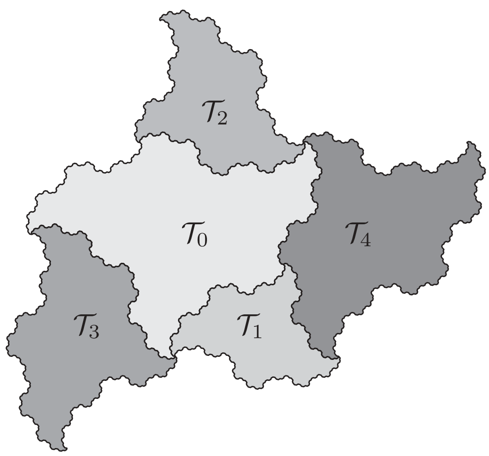

FIGURE 3. *The decomposition of the central tile associated with η into subtiles. According to the expansion $d_\eta(1) = 10001$, the central tile is subdivided into exactly five subtiles.*

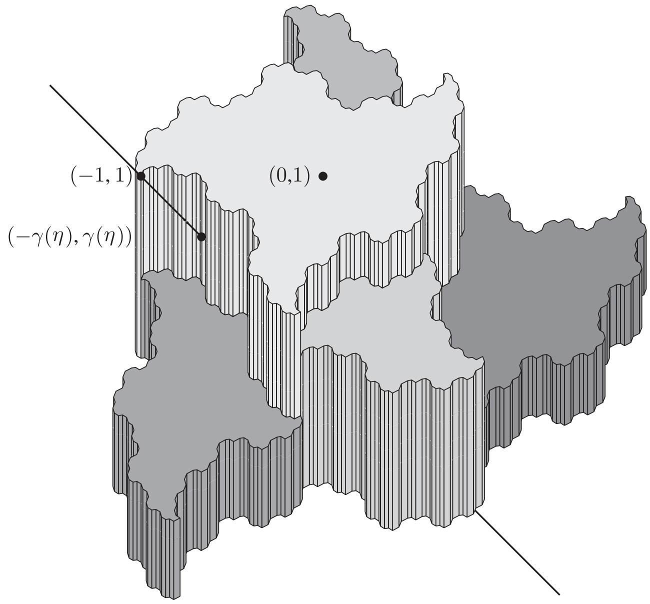

FIGURE 4. *The set $\mathcal{E}_\eta$ and the line $(-x, x)$, with $x \in \mathbb{R}$.*

The subdivision of the central tile into the subtiles $\mathcal{T}_i$ allows us to characterize those real numbers in $(0, 1)$ having a purely periodic β-expansion. More precisely, Ito and Rao [**16**] proved the following result (see also [**10**] for a shorter and more natural proof).

THEOREM IR. *Let β be a Pisot unit and let $x \in [0, 1)$. The β-expansion of x is purely periodic if and only if $x \in \mathbb{Q}(\beta)$ and*

$$(-\Xi(x), x) \in \mathcal{E}_\beta := \bigcup_{i=0}^{n+m-1} \mathcal{T}_i \times [0, T_\beta^i(1)).$$

We immediately deduce the following result from Theorem IR (see Figure 4).

COROLLARY 4.2. *Let β be a Pisot unit. Then one of the following holds:*

(i) $\gamma(\beta) = T_\beta^i(1)$ *for some* $i \in \{0, \ldots, n + m - 1\}$;

(ii) *the* $(r + s - 1)$*-dimensional vector* $(-\gamma(\beta), \ldots, -\gamma(\beta))$ *is in* $\mathcal{T}_i \cap \mathcal{T}_j$ *with* $T_\beta^j(1) < \gamma(\beta) < T_\beta^i(1)$;

(iii) *the* $(r + s - 1)$*-dimensional vector* $(-\gamma(\beta), \ldots, -\gamma(\beta))$ *lies on the boundary of* $\mathcal{T}$.

5. *Some results on $\Xi(\beta)$-representations*

In this section, we consider representations of points in the space $\mathbb{R}^{r-1} \times \mathbb{C}^s$ that involve some Galois conjugations related to β. Such representations are termed $\Xi(\beta)$-representations. If $x = (x_1, \ldots, x_{r+s-1})$ and $z = (y_1, \ldots, y_{r+s-1})$ are two elements in $\mathbb{R}^{r-1} \times \mathbb{C}^s$, then we set $x \odot z := (x_1 y_1, \ldots, x_{r+s-1} y_{r+s-1})$. From now on, we say that a pointed bi-infinite word $\ldots a_{-1} a_0{}^\bullet a_1 a_2 \ldots$ is admissible if $\ldots a_{-1} a_0 a_1 a_2 \ldots$ is a bi-infinite admissible word.

Definition 5.1. A $\Xi(\beta)$-*representation* of $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ is an admissible bi-infinite word $\ldots a_{-1}a_0{}^\bullet a_1 a_2 \ldots$ with ${\scriptstyle\bullet} a_1 a_2 \ldots \in \mathbb{Z}[\beta]$ and such that

$$z = \sum_{j=0}^{\infty} a_{-j}\Xi(\beta^j) + \Xi({\scriptstyle\bullet} a_1 a_2 \ldots).$$

We derive now several results about $\Xi(\beta)$-representations that will be useful in what follows. First, we show that such representations do exist.

Lemma 5.2. *Every $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ has at least one $\Xi(\beta)$-representation.*

Proof. As we already mentioned, we have

$$\bigcup_{y \in \mathbb{Z}[\beta] \cap [0,1)} \mathcal{T}(y) = \mathbb{R}^{r-1} \times \mathbb{C}^s.$$

Thus every $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ belongs to some tile $\mathcal{T}(y)$ with $y \in \mathbb{Z}[\beta]$. By the definition of $\mathcal{T}(y)$, there exists a sequence of finite words W_k such that $W_k d_\beta(y)$ is an admissible infinite word and

$$\lim_{k \to \infty} \Xi(W_k {\scriptstyle\bullet}) = z - \Xi(y).$$

Now, note that there exist infinitely many W_k that end with the same letter, say a_0. Among them, there are infinitely many of them with the same last but one letter, say a_{-1}. Keeping on this procedure, we deduce the existence of a left infinite word $\ldots a_{-1}a_0$ such that $\ldots a_{-1}a_0{}^\bullet d_\beta(y)$ is a bi-infinite admissible word and

$$z - \Xi(y) = \lim_{k \to \infty} \Xi(a_{k+1} \ldots a_0 {\scriptstyle\bullet}).$$

Thus, we have $z = \sum_{j=0}^{+\infty} a_{-j}\Xi(\beta^j) + \Xi(y)$, which has proved that $\ldots a_{-1}a_0{}^\bullet d_\beta(y)$ is a $\Xi(\beta)$-representation of z since y belongs to $\mathbb{Z}[\beta]$. This ends the proof. □

The following results were shown by Sadahiro [**21**] for cubic Pisot units β satisfying (F) with a single pair of complex Galois conjugates, that is $r = s = 1$.

Lemma 5.3. *There exists a positive integer C such that every $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ has at most C different $\Xi(\beta)$-representations.*

Proof. We already observed in Fact 3.3 that there exists a positive integer, say C, such that every $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ is contained in at most C different tiles $\mathcal{T}(y)$. Let $z \in \mathbb{R}^{r-1} \times \mathbb{C}^s$ and let us assume that z has more than C different $\Xi(\beta)$-representations, namely

$$(\ldots a_{-1}^{(j)} a_0^{(j)\bullet} a_1^{(j)} a_2^{(j)} \ldots)_{1 \leqslant j \leqslant C+1}.$$

Then, there exists some non-negative integer k such that the infinite sequences $a_{-k+1}^{(j)} a_{-k+2}^{(j)} \cdots$, with $1 \leqslant j \leqslant C+1$, are all distinct. This implies that $\Xi(\beta^k) \odot z$ belongs to each tile $\mathcal{T}(y_j)$ with $y_j = {\scriptstyle\bullet} a_{-k+1}^{(j)} a_{-k+2}^{(j)} \ldots$. Consequently, $\Xi(\beta^k) \odot z$ lies in more than C different tiles, which contradicts the definition of C. □

Lemma 5.4. *Let $\ldots a_{-1}a_0{}^\bullet a_1 a_2 \ldots$ be a $\Xi(\beta)$-representation of $\Xi(x)$ for some $x \in \mathbb{Q}(\beta)$. Then the left infinite word $\ldots a_{-1}a_0$ is eventually periodic.*

Proof. Let $\dots a_{-1}a_0{}^{\bullet}a_1a_2\dots$ be a $\Xi(\beta)$-representation of $\Xi(x)$ for some $x \in \mathbb{Q}(\beta)$. Set $x_k := a_{-k+1}\dots a_0{\bullet}a_1a_2\dots$ and for every non-negative integer k set $z_k := \beta^{-k}(x - x_k)$. Then, we have

$$z_{k+1} = \frac{1}{\beta}(z_k - a_{-k}),$$

which implies that the set $\{z_k \mid k \geqslant 0\}$ is bounded. Furthermore, we have

$$\Xi(z_k) = \sum_{j=0}^{\infty} a_{-k-j}\Xi(\beta^j).$$

Since β is a Pisot number, this implies that all conjugates of z_k are bounded as well. Since all the z_k have a degree bounded by the degree of β, this implies that $\{z_k \mid k \geqslant 0\}$ is a finite set.

Now observe that $\dots a_{-k-1}a_{-k}{}^{\bullet}0^{\omega}$ is a $\Xi(\beta)$-representation of $\Xi(z_k)$. By Lemma 5.3, each $\Xi(z_k)$ has at most C different $\Xi(\beta)$-representations. Since there are only finitely many different z_k, the set of left infinite words $\{\dots a_{-k-1}a_{-k} \mid k \geqslant 0\}$ is finite. This implies that the left infinite word $\dots a_{-1}a_0$ is eventually periodic, concluding the proof. □

6. *Spiral points*

In this section, we introduce a topological and geometrical notion for compact subsets of the complex plane. This notion of *spiral point* turns out to be the key tool for proving Theorems 1.1 and 1.2. Roughly, x is a spiral point with respect to a compact set $X \subset \mathbb{C}$ when both the interior and the complement of X turn around x, meaning that they meet infinitely many times all rays of positive length issued from x. More formally, we have the following definition.

DEFINITION 6.1. Let X be a compact subset of $\mathbb{C}$. A point $z \in X$ is a *spiral point* with respect to X if for all positive real numbers ε and θ, both the interior of X and the complement of X meet the ray $z + [0, \varepsilon)e^{i\theta} := \{z + \rho e^{i\theta} \mid \rho \in [0, \varepsilon)\}$.

It seems that the boundary of many fractal objects in the complex plane contains some spiral points, but we were not able to find a reference for this notion. The most common property studied in fractal geometry that is related to our notion of spiral point seems to be the non-existence of weak tangent (see, for instance, [**12**]). For instance, if X denotes a set with a non-integer Hausdorff dimension lying between 1 and 2, then almost all points of X do not have a weak tangent. This result applies in particular to the boundary of some classical fractal structures such as Julia sets and Heighway dragon.

Our key result now reads as follows (see Figure 5).

PROPOSITION 6.2. *Let β be a cubic Pisot number with a complex Galois conjugate α. Then every point in $\mathbb{Q}(\alpha)$ that belongs to the boundary of $\mathcal{T}$ or of a subtile $\mathcal{T}_i$ is a spiral point with respect to this tile.*

In order to prove Proposition 6.2, we will need the following result.

LEMMA 6.3. *Let β be a Pisot number with complex Galois conjugates $\beta_j, \overline{\beta_j}$, $r < j \leqslant r+s$, $\beta_j = \rho_j e^{2\pi i\phi_j}$. Then $1, \phi_{r+1}, \dots, \phi_{r+s}$ are linearly independent over $\mathbb{Q}$.*

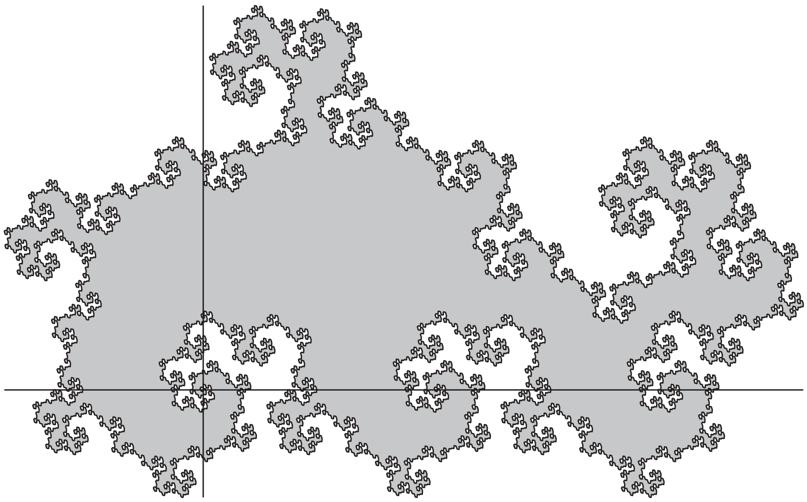

FIGURE 5. *The central tile associated with the real root of* $x^3 - 3x^2 + 2x - 1$.

Proof. Let $k_0, \ldots, k_s$ be integers such that $k_0 + k_1\phi_{r+1} + \ldots + k_s\phi_{r+s} = 0$. Then the product $\beta_{r+1}^{k_1} \ldots \beta_{r+s}^{k_s}$ is a real number and thus

$$\beta_{r+1}^{k_1} \ldots \beta_{r+s}^{k_s} = \overline{\beta_{r+1}}^{k_1} \ldots \overline{\beta_{r+s}}^{k_s}.$$

It is proved in [**17**] that this implies that $k_1 = \ldots = k_s = 0$, and the lemma is proved. □

We are now ready to prove Proposition 6.2.

Proof of Proposition 6.2. First note that since β is a cubic Pisot number, we simply have $\Xi(\beta) = \alpha$. Let $z \in \mathbb{Q}(\alpha) \cap \mathcal{T}$ and let θ and ε be two positive real numbers.

By Lemma 5.2, we see that z has at least one α-representation. Since z belongs to the central tile $\mathcal{T}$, the proof of Lemma 5.2 actually implies the existence of a left infinite admissible word $\ldots a_{-1}a_0$ such that $\ldots a_{-1}a_0{\bullet}0^\omega$ is an α-representation of z. Furthermore, by Lemma 5.4, such a representation is eventually periodic and there thus exist non-negative integers p and q such that ${}^\omega(a_{-q-p+1} \ldots a_{-q})a_{-q+1} \ldots a_0{\bullet}0^\omega$ is an α-representation of z.

For every non-negative integer k, set $z_k := (a_{-q-p+1} \ldots a_{-q})^k a_{-q+1} \ldots a_0{\bullet}$. There exists a positive integer ℓ (depending on β) such that, for every left infinite admissible word $\ldots b_{-1}b_0$, the bi-infinite word

$$\ldots b_{-1}b_0 0^\ell (a_{-q-p+1} \ldots a_{-q})^j a_{-q+1} \ldots a_0{\bullet}0^\omega \tag{6.1}$$

is also admissible. Roughly, this means that the lexicographic condition cannot 'jump over 0^ℓ'. Consequently, we have

$$z_k + \alpha^{q+kp+\ell}\mathcal{T} \subseteq \mathcal{T}. \tag{6.2}$$

Since β is a Pisot unit, we know that $\mathcal{T}$ has a non-empty interior, a result obtained in [**5**]. Thus, $\mathcal{T}$ contains some ball, say $\mathcal{B}$. Set $\mathcal{B}' := \alpha^{q+\ell}\mathcal{B}$. By (6.2), it also contains the balls $z_k + \alpha^{kp}\mathcal{B}'$ for every non-negative integer k.

Note that there exists some non-empty interval $(\eta, \zeta) \subset (0, 1)$ and some positive real number R such that every ray $z + [0, R)e^{2\pi i\psi}$ with $\psi \in (\eta, \zeta)$ contains an interior point of $z_0 + \mathcal{B}'$. Furthermore, ${}^\omega(a_{-q-p+1} \ldots a_{-q})0^{q+kp}{\bullet}0^\omega$ is an α-representation of $z - z_k$ and thus

$$\alpha^{kp}\mathcal{B}' + z_k - z = \alpha^{kp}(\mathcal{B}' + z_0 - z).$$

Let ρ and ϕ be positive real numbers such that $\alpha = \rho e^{2\pi i\phi}$. Then every ray

$$z + [0, \rho^{kp} R) e^{2\pi i(\psi + kp\phi)}$$

with $\psi \in (\eta, \zeta)$ contains an interior point of $z_k + \alpha^{kp}\mathcal{B}'$. By Lemma 6.3, we see that ϕ is irrational and the sequence $(kp\phi \bmod 1)_{k\geqslant 0}$ is thus dense in $(0, 1)$. It follows that there are infinitely many positive integers $k_1 < k_2 < \dots$ and infinitely many real numbers $x_1, x_2, \dots \in (\eta, \zeta)$ such that $\theta/2\pi = k_h p\phi + x_h \bmod 1$ for $h \geqslant 1$. Since β is a Pisot number, we have $0 < \rho < 1$. For ℓ large enough, we thus obtain that the ray $z + [0, \varepsilon)e^{i\theta}$ contains an interior point of $\mathcal{T}$.

If z belongs to the boundary of $\mathcal{T}$, from the covering property (3.1) it follows that z is also contained in some tile $\mathcal{T}(y)$ with $y \neq 0$.

Then, arguing as previously, we obtain that z has an α-representation of the form

$$^{\omega}(a'_{-q'-p'+1} \dots a'_{-q'}) a'_{-q'+1} \dots a'_0 {}^{\bullet} d_\beta(y)$$

and by similar arguments as above, we can show that the ray $z + [0, \varepsilon)e^{i\theta}$ contains an interior point of the tile $\mathcal{T}(y)$. By Theorem ARS, such a point lies in the complement of $\mathcal{T}$. This shows that z is a spiral point with respect to $\mathcal{T}$.

We shall now detail why similar arguments apply if we replace $\mathcal{T}$ by a subtile $\mathcal{T}_j$. Recall that $d^*_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+n})^\omega$ denotes the expansion of 1. From the definition of the subtiles $\mathcal{T}_j$ it follows that:

(i) when $0 \leqslant j < m$, a point z belongs to $\mathcal{T}_j$ if and only if z has an α-representation with $t_1 \dots t_j {}^{\bullet} 0^\omega$ as a suffix;

(ii) when $m \leqslant j < m + n$, a point z belongs to $\mathcal{T}_j$ if and only if there exists $\ell \geqslant 0$ such that z has an α-representation with $t_1 \dots t_m (t_{m+1} \dots t_{m+n})^\ell t_{m+1} \dots t_j {}^{\bullet} 0^\omega$ as a suffix.

Let us assume that $z \in \mathbb{Q}(\alpha) \cap \partial\mathcal{T}_j$ for some $0 \leqslant j < m + n$. If z also belongs to ∂T, we fall into the previous case. We can thus assume that this is not the case, and hence z belongs to the boundary of another tile $\mathcal{T}_h$, with $h \neq j$. By Proposition BS, it remains to prove that the ray $z + [0, \varepsilon)e^{i\theta}$ contains an interior point of both $\mathcal{T}_j$ and $\mathcal{T}_h$.

Let us briefly justify this claim. Coming back to the proof above, we deduce from (i) and (ii) that z_k belongs to $\mathcal{T}_j$ for k large enough. Moreover, there exists a positive integer ℓ (depending on β) such that, for every left infinite admissible word $\dots b_{-1} b_0$, the bi-infinite word

$$\dots b_{-1} b_0 0^\ell (a_{-q-p+1} \dots a_{-q})^k a_{-q+1} \dots a_0 {}^{\bullet} 0^\omega$$

satisfies the admissibility condition for $\mathcal{T}_j$ given in (i)/(ii).

This allows us to replace (6.2) by

$$z_k + \alpha^{q+kp+\ell} \mathcal{T} \subseteq \mathcal{T}_j$$

and to conclude as previously. □

7. *Proof of Theorems* 1.1 *and* 1.2

In this section, we complete the proof of Theorems 1.1 and 1.2. In this section, β denotes a cubic Pisot unit.

LEMMA 7.1. *For every $i \geqslant 1$, either $T^i_\beta(1) = 0$ or $T^i_\beta(1) \notin \mathbb{Q}$.*

Proof. This follows from $T^i_\beta(1) \in \mathbb{Z}[\beta] \cap [0, 1)$. □

LEMMA 7.2. *If β satisfies* (F), *then $\Xi(-1)$ lies on the boundary of $\mathcal{T}$.*

Proof. If β satisfies (F), then we have $d_\beta(1) = t_1 \dots t_n 0^\omega$, with $t_n > 0$. If $1 \leqslant j \leqslant n$ is an integer such that $t_j > 0$, then

$$^\omega(t_1 \dots t_{n-1}(t_n - 1))t_1 \dots t_{j-1}(t_j - 1)^\bullet t_{j+1} \dots t_n 0^\omega$$

is a $\Xi(\beta)$-expansion of -1 since this sequence is admissible and

$$\lim_{k\to\infty} \Xi\big((t_1 \dots t_{n-1}(t_n - 1))^k t_1 \dots t_j {\bullet} t_{j+1} \dots t_n\big) = \lim_{k\to\infty} \Xi(\beta^{j+kn}) = 0.$$

Since by definition $t_n > 0$, it follows that $\Xi(-1)$ belongs to the central tile $\mathcal{T}$. Since $t_1 > 0$, it follows that $\Xi(-1)$ also lies in the tile $\mathcal{T}(y)$ where $y := {\bullet}t_2 \dots t_n \neq 0$. By Theorem ARS, we obtain that $\Xi(-1)$ lies on the boundary of the tile $\mathcal{T}$, as claimed. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. As previously mentioned, if β satisfies (F), we infer from [**2**] that $\gamma(\beta) > 0$.

If β does not satisfy (F), then Akiyama [**4**] proved that the minimal polynomial $p(x)$ of β satisfies either $p(0) = 1$ or $p(x) = x^3 - ax^2 - bx - 1$ with $-a + 1 \leqslant b \leqslant -2$. Hence, β has either a positive real Galois conjugate or two complex Galois conjugates. In the first case, it is easy to see that $\gamma(\beta) = 0$, as observed in [**2**]. In the latter case, it is known that the origin belongs to the boundary of the central tile $\mathcal{T}$. By Proposition 6.2, we get that 0 is a spiral point with respect to $\mathcal{T}$. Hence, there are rational numbers arbitrarily close to 0 which have a β-expansion that is not purely periodic (as well as intervals where all rational numbers have purely periodic β-expansion). Consequently, we also have $\gamma(\beta) = 0$ in that case, concluding the proof. □

Proof of Theorem 1.2. Let β be a cubic Pisot unit satisfying (F). Let us assume that $\mathbb{Q}(\beta)$ is not a totally real number field. Let us assume that $\gamma(\beta)$ is a rational number and we aim at deriving a contradiction. Note that by assumption we have $\Xi(-\gamma(\beta)) = -\gamma(\beta)$.

We first observe that if $\gamma(\beta) = T_\beta^i(1)$ for some non-negative integer i, then $-\gamma(\beta)$ belongs to the boundary of the central tile $\mathcal{T}$. Indeed, in that case, Lemma 7.1 implies that $\gamma(\beta) = T^0(1) = 1$ and the result follows from Lemma 7.2.

Let us assume that $-\gamma(\beta)$ belongs to the boundary of the central tile $\mathcal{T}$. By Proposition 6.2, $-\gamma(\beta)$ is a spiral point with respect to $\mathcal{T}$. Thus there exists a rational number $0 < r < \gamma(\beta)$ such that $-r$ lies in the interior of a tile $\mathcal{T}(y)$ for some $y \neq 0$. By Theorem ARS, $-r = \Xi(-r)$ does not belong to $\mathcal{T}$ and thus

$$(-r, r) \notin \bigcup_{i=0}^{m+n-1} \mathcal{T}_i \times [0, T_\beta^i(1)).$$

Theorem IR then implies that $d_\beta(r)$ is not purely periodic, which contradicts the definition of $\gamma(\beta)$ since $0 < r < \gamma(\beta)$.

Let us assume now that $-\gamma(\beta)$ does not belong to the boundary of the central tile $\mathcal{T}$. By Corollary 4.2 and our first observation, this ensures the existence of two integers $0 \leqslant i \neq j \leqslant m + n - 1$ such that $-\gamma(\beta) \in \mathcal{T}_i \cap \mathcal{T}_j$ and $T_\beta^j(1) < \gamma(\beta) < T_\beta^i(1)$. By Proposition BS, we get that $-\gamma(\beta) \in \partial\mathcal{T}_j$. We then infer from Proposition 6.2 that $-\gamma(\beta)$ is a spiral point with respect to $\mathcal{T}_j$. The interior of $\mathcal{T}_j$ thus contains a rational number $-r$ such that $T_\beta^j(1) < r < \gamma(\beta)$. By Proposition BS, $-r = \Xi(-r)$ does not belong to any other subtile $\mathcal{T}_i$ and thus

$$(-r, r) \notin \bigcup_{i=0}^{m+n-1} \mathcal{T}_i \times [0, T_\beta^i(1)).$$

By Theorem IR, the β-expansion of r is not purely periodic, which yields a contradiction with the fact that $0 < r < \gamma(\beta)$. This concludes the proof. □

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