

# On the complexity of algebraic numbers, II. Continued fractions

by

BORIS ADAMCZEWSKI

and

YANN BUGEAUD

*CNRS & Université Claude Bernard  
Villeurbanne, France*

*Université Louis Pasteur  
Strasbourg, France*

## 1. Introduction

Let  $b \geq 2$  be an integer. Émile Borel [9] conjectured that every real irrational algebraic number  $\alpha$  should satisfy some of the laws shared by almost all real numbers with respect to their  $b$ -adic expansions. Despite some recent progress [1], [3], [7], we are still very far away from establishing such a strong result. In the present work, we are concerned with a similar question, where the  $b$ -adic expansion of  $\alpha$  is replaced by its sequence of partial quotients. Recall that the continued fraction expansion of an irrational number  $\alpha$  is eventually periodic if and only if  $\alpha$  is a quadratic irrationality. However, very little is known regarding the size of the partial quotients of algebraic real numbers of degree at least three. Because of some numerical evidence and a belief that these numbers behave like most of the numbers in this respect, it is often conjectured that their partial quotients form an unbounded sequence, but we seem to be very far away from a proof (or a disproof). Apparently, Khinchin [16] was the first to consider such a question (see [4], [27] and [29] for surveys including a discussion on this problem). Although almost nothing has been proved yet in this direction, some more general speculations are due to Lang [17], including the fact that algebraic numbers of degree at least three should behave like most of the numbers with respect to the Gauss–Khinchin–Kuzmin–Lévy laws.

More modestly, we may expect that if the sequence of partial quotients of an irrational number  $\alpha$  is, in some sense, ‘simple’, then  $\alpha$  is either quadratic or transcendental. The term ‘simple’ can of course lead to many interpretations. It may denote real numbers whose continued fraction expansion has some regularity, or can be produced by a simple algorithm (by a simple Turing machine, for example), or arises from a simple dynamical system ... The main results of the present work are two new combinatorial transcen-

dence criteria, which considerably improve upon those from [5], [13] and [8]. It is of a particular interest that such criteria naturally yield, in a unified way, several new results on the different approaches of the above-mentioned notion of simplicity/complexity for the continued fraction expansions of algebraic real numbers of degree at least three.

This article is organized as follows. §2 is devoted to the statements of our two transcendence criteria. Several applications of them are then briefly discussed in §3. All the proofs are postponed to §§4 and 5.

## 2. Transcendence criteria for stammering continued fractions

Before stating our theorems, we need to introduce some notation. Let  $\mathcal{A}$  be a given set, not necessarily finite. The length of a word  $W$  on the alphabet  $\mathcal{A}$ , that is, the number of letters composing  $W$ , is denoted by  $|W|$ . For positive integers  $l$ , we write  $W^l$  for the word  $W \dots W$  ( $l$  times repeated concatenation of the word  $W$ ). More generally, for positive rational numbers  $x$ , we denote by  $W^x$  the word  $W^{[x]}W'$ , where  $W'$  is the prefix of  $W$  of length  $\lceil (x-[x])|W| \rceil$ . Here, and in all what follows,  $[y]$  and  $\lceil y \rceil$  denote, respectively, the integer part and the upper integer part of the real number  $y$ . Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of elements from  $\mathcal{A}$ , that we identify with the infinite word  $a_1 a_2 \dots a_l \dots$ . Let  $w$  be a rational number with  $w > 1$ . We say that  $\mathbf{a}$  satisfies Condition  $(*)_w$  if  $\mathbf{a}$  is not eventually periodic and if there exists a sequence of finite words  $\{V_n\}_{n \geq 1}$  such that

- (i) for  $n \geq 1$ , the word  $V_n^w$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) the sequence  $\{|V_n|\}_{n \geq 1}$  is increasing.

Roughly speaking,  $\mathbf{a}$  satisfies Condition  $(*)_w$  if  $\mathbf{a}$  is not eventually periodic and if there exist infinitely many ‘non-trivial’ repetitions (the size of which is measured by  $w$ ) at the beginning of the infinite word  $a_1 a_2 \dots a_l \dots$ .

Our transcendence criterion for ‘purely’ stammering continued fractions can be stated as follows:

**THEOREM 1.** *Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_l, \dots].$$

*If there exists a rational number  $w \geq 2$  such that  $\mathbf{a}$  satisfies Condition  $(*)_w$ , then  $\alpha$  is transcendental. If there exists a rational number  $w > 1$  such that  $\mathbf{a}$  satisfies Condition  $(*)_w$ , and if the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  is bounded (which is, in particular, the case when the sequence  $\mathbf{a}$  is bounded), then  $\alpha$  is transcendental.*

The main interest of the first statement of Theorem 1 is that there is no condition on the growth of the sequence  $\{q_l\}_{l \geq 1}$ . Apparently, this fact has not been observed

previously. The second statement of Theorem 1 improves upon Theorem 4 from [5], which requires, together with some extra rather constraining hypotheses, the stronger assumption  $w > \frac{3}{2}$ . The condition that the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  has to be bounded is in general very easy to check, and is not very restrictive, since it is satisfied by almost all real numbers (in the sense of the Lebesgue measure). Apart from this assumption, Theorem 1 does not depend on the size of the partial quotients of  $\alpha$ . This is in a striking contrast to all previous results [5], [13], [8], in which, roughly speaking, the size  $w$  of the repetition is required to be larger if the partial quotients are larger. Unlike these results, our Theorem 1 can be easily applied even if  $\alpha$  has unbounded partial quotients.

Unfortunately, in the statement of Theorem 1, the repetitions must appear at the very beginning of  $\mathbf{a}$ . Results from [13] allow a shift, whose length, however, must be controlled in terms of the size of the repetitions. Similar results cannot be deduced from our Theorem 1. However, many ideas from the proof of Theorem 1 can be used to deal also with this situation, under some extra assumptions, and to improve upon the transcendence criterion from [13].

Keep the notation introduced at the beginning of this section. Let  $w$  and  $w'$  be non-negative rational numbers with  $w > 1$ . We say that  $\mathbf{a}$  satisfies Condition  $(**)_{w,w'}$  if  $\mathbf{a}$  is not eventually periodic and if there exist two sequences of finite words  $\{U_n\}_{n \geq 1}$  and  $\{V_n\}_{n \geq 1}$  such that

- (i) for  $n \geq 1$ , the word  $U_n V_n^w$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) the sequence  $\{|U_n|/|V_n|\}_{n \geq 1}$  is bounded from above by  $w'$ ;
- (iii) the sequence  $\{|V_n|\}_{n \geq 1}$  is increasing.

We are now ready to state our transcendence criterion for (general) stammering continued fractions.

**THEOREM 2.** *Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_l, \dots].$$

*Assume that the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  is bounded and set  $M = \limsup_{l \rightarrow +\infty} q_l^{1/l}$  and  $m = \liminf_{l \rightarrow +\infty} q_l^{1/l}$ . Let  $w$  and  $w'$  be non-negative real numbers with*

$$w > (2w' + 1) \frac{\log M}{\log m} - w'. \quad (1)$$

*If  $\mathbf{a}$  satisfies Condition  $(**)_{w,w'}$ , then  $\alpha$  is transcendental.*

We display an immediate consequence of Theorem 2.

COROLLARY 1. Let  $\mathbf{a}=\{a_l\}_{l\geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l\geq 1}$  denote the sequence of convergents to the real number

$$\alpha := [0; a_1, a_2, \dots, a_l, \dots].$$

Assume that the sequence  $\{q_l^{1/l}\}_{l\geq 1}$  converges. Let  $w$  and  $w'$  be non-negative real numbers with  $w > w' + 1$ . If  $\mathbf{a}$  satisfies Condition  $(**)_{w,w'}$ , then  $\alpha$  is transcendental.

Our Theorem 2 improves Theorem 6.3 of Davison [13]. Indeed, to apply his transcendence criterion,  $w$  and  $w'$  must satisfy

$$w > \left(2w' + \frac{3}{2}\right) \frac{\log M}{\log m},$$

which is a far stronger condition than (1).

Theorems 1 and 2 yield many new results that could not be obtained with the earlier transcendence criteria. Some of them are stated in §3, while many others will be given in a subsequent work [2]. Theorems 1 and 2 are in the same spirit as the following result, established in [1] and [3], and which deals with the transcendence of  $b$ -adic expansions.

THEOREM ABL. Let  $b \geq 2$  be an integer. Let  $\mathbf{a}=\{a_l\}_{l\geq 1}$  be a sequence of integers in  $\{0, \dots, b-1\}$ . Let  $w$  and  $w'$  be non-negative rational numbers with  $w > 1$ . If  $\mathbf{a}$  satisfies Condition  $(**)_{w,w'}$ , then the real number  $\sum_{l\geq 1} a_l/b^l$  is transcendental.

Theorem ABL is as strong for ‘purely’ stammering sequences as for general stammering sequences, provided that the repetitions do not occur too far away from the beginning of the infinite word. Unfortunately, in Theorem 2 we are unable to replace the assumption ‘ $w > w' + 1$ ’ by the weaker one ‘ $w > 1$ ’, occurring in Theorem ABL.

The main tool for the proofs of Theorems 1 and 2, given in §4, is the Schmidt subspace theorem [25], [26]. This (more precisely, a  $p$ -adic version of it) is also the key auxiliary result for establishing Theorem ABL.

### 3. Applications to the complexity of algebraic continued fractions

Our transcendence criteria apply to establish that several well-known continued fractions are transcendental, including the Thue–Morse continued fraction (whose transcendence was first proved by M. Queffélec [21]), the Rudin–Shapiro continued fraction, folded continued fractions, continued fractions arising from perturbed symmetries (these sequences were introduced by Mendès France [18]), continued fractions considered by Davison [13] and Baxa [8], etc. These applications are discussed in details in [2], where complete proofs are given. We only focus here on applications related to our main problem, that is, to the complexity of algebraic numbers with respect to their continued fraction expansions.

### 3.1. An algorithmic approach

We first briefly discuss how the complexity of the continued fraction of real numbers can be interpreted in an algorithmic way. Following the pioneering work of Turing [28], a sequence is said to be computable if there exists a Turing machine capable to produce its terms successively. Later, Hartmanis and Stearns [15] proposed to emphasize the quantitative aspect of this notion, and to take into account the number  $T(n)$  of operations needed by a (multitape) Turing machine to produce the first  $n$  terms of the sequence. In this regard, a real number is considered to be simpler if its continued fraction expansion can be produced faster by a Turing machine.

Finite automata are one of the most basic models of computation and take thus place at the bottom of the hierarchy of Turing machines. In particular, such machines produce sequences in real time, that is, with  $T(n)=O(n)$ . An infinite sequence  $\mathbf{a}=\{a_n\}_{n\geq 0}$  is said to be generated by a  $k$ -automaton if  $a_n$  is a finite-state function of the base- $k$  representation of  $n$ . This means that there exists a finite automaton starting with the  $k$ -ary expansion of  $n$  as input and producing the term  $a_n$  as output. A nice reference on this topic is the book of Allouche and Shallit [6]. As a classical example of a sequence generated by a 2-automaton, we mention the famous binary Thue–Morse sequence  $\mathbf{a}=\{a_n\}_{n\geq 0}=0110100110010\dots$ . This sequence is defined as follows:  $a_n$  is equal to 0 (resp. to 1) if the sum of the digits in the binary expansion of  $n$  is even (resp. is odd). In view of the above discussion, we may expect that finite automata are ‘too simple’ Turing machines to produce the continued fraction expansion of algebraic numbers that are neither rationals nor quadratics.

*Problem 1.* Do there exist algebraic numbers of degree at least three whose continued fraction expansion can be produced by a finite automaton?

Thanks to Cobham [11], we know that sequences generated by finite automata can be characterized in terms of iterations of morphisms of free monoids generated by finite sets. We now recall this useful description. For a finite set  $\mathcal{A}$ , let  $\mathcal{A}^*$  denote the free monoid generated by  $\mathcal{A}$ . The empty word is the neutral element of  $\mathcal{A}^*$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be two finite sets. An application from  $\mathcal{A}$  to  $\mathcal{B}^*$  can be uniquely extended to a homomorphism between the free monoids  $\mathcal{A}^*$  and  $\mathcal{B}^*$ . Such a homomorphism is called a *morphism from  $\mathcal{A}$  to  $\mathcal{B}$* . If there is a positive integer  $k$  such that each element of  $\mathcal{A}$  is mapped to a word of length  $k$ , then the morphism is called  *$k$ -uniform* or simply *uniform*. Similarly, an application from  $\mathcal{A}$  to  $\mathcal{B}$  can be uniquely extended to a homomorphism between the free monoids  $\mathcal{A}^*$  and  $\mathcal{B}^*$ . Such an application is called a *coding* (the term ‘letter-to-letter’ morphism is also used in the literature).

A morphism  $\sigma$  from  $\mathcal{A}$  into itself is said to be *prolongable* if there exists a letter  $a$

such that  $\sigma(a)=aW$ , where the word  $W$  is such that  $\sigma^n(W)$  is a non-empty word for every  $n \geq 0$ . In that case, the sequence of finite words  $\{\sigma^n(a)\}_{n \geq 1}$  converges in  $\mathcal{A}^{\mathbb{Z}_{\geq 0}}$  (endowed with the product topology of the discrete topology on each copy of  $\mathcal{A}$ ) to an infinite word  $\mathbf{a}$ . This infinite word is clearly a fixed point for  $\sigma$ . We say that a sequence  $\mathbf{b}$  is generated by the morphism  $\sigma$  if there exists a coding  $\varphi$  such that  $\mathbf{b}=\varphi(\mathbf{a})$ . If, moreover, every letter appearing in  $\mathbf{a}$  occurs at least twice, then we say that  $\mathbf{b}$  is generated by a recurrent morphism. If the alphabet  $\mathcal{A}$  has only two letters, then we say that  $\mathbf{b}$  is generated by a binary morphism. Furthermore, if  $\sigma$  is uniform, then we say that  $\mathbf{b}$  is generated by a uniform morphism.

For instance, the Fibonacci morphism  $\sigma$  defined on the alphabet  $\{0, 1\}$  by  $\sigma(0)=01$  and  $\sigma(1)=0$  is a binary, recurrent and non-uniform morphism which generates the celebrated Fibonacci infinite word

$$\mathbf{a} = \lim_{n \rightarrow +\infty} \sigma^n(0) = 010010100100101001 \dots$$

Uniform morphisms and automatic sequences are strongly connected, as shown by the following result of Cobham [11]:

**THEOREM (Cobham).** *A sequence can be generated by a finite automaton if and only if it is generated by a uniform morphism.*

This useful description gives rise to the following challenging question:

*Problem 2.* Do there exist algebraic numbers of degree at least three whose continued fraction expansion is generated by a morphism?

Our main contribution towards both problems is the following result:

**THEOREM 3.** *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a recurrent morphism.*

The class of primitive morphisms has been extensively studied. In particular, Theorem 3 fully solved a question studied by M. Queffélec [22]. We display the following direct consequence of Theorem 3:

**COROLLARY 2.** *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a binary morphism.*

Indeed, it is easy to see that binary morphisms are either recurrent or they generate only eventually periodic sequences.

### 3.2. A dynamical approach

In this section, we discuss the notion of complexity of the continued fraction expansion of a real number from a dynamical point of view.

Let  $\mathcal{A}$  be a given set, finite or not. A subshift on  $\mathcal{A}$  is a symbolic dynamical system  $(X, S)$ , where  $S$  is the classical shift transformation defined from  $\mathcal{A}^{\mathbf{Z}_{\geq 1}}$  into itself by  $S(\{a_n\}_{n \geq 1}) = \{a_n\}_{n \geq 2}$  and  $X$  is a subset of  $\mathcal{A}^{\mathbf{Z}_{\geq 1}}$  such that  $S(X) \subset X$ . With an infinite sequence  $\mathbf{a}$  in  $\mathcal{A}^{\mathbf{Z}_{\geq 1}}$ , we associate the subshift  $\mathcal{X}_{\mathbf{a}} = (X, S)$ , where  $X := \overline{\mathcal{O}(\mathbf{a})}$  denotes the closure of the orbit of the sequence  $\mathbf{a}$  under the action of  $S$ . The complexity function  $p_{\mathbf{a}}$  of a sequence  $\mathbf{a}$  associates with a positive integer  $n$  the number  $p_{\mathbf{a}}(n)$  of distinct blocks of  $n$  consecutive letters occurring in it. More generally, the complexity function  $p_{\mathcal{X}}$  of a subshift  $\mathcal{X} = (X, S)$  associates with a positive integer  $n$  the number  $p_{\mathcal{X}}(n)$  of distinct blocks of  $n$  consecutive letters occurring in at least one element of  $X$ .

With a subshift  $\mathcal{X} = (X, S)$  on  $\mathbf{Z}_{\geq 1}$  one can associate the set  $\mathcal{C}_{\mathcal{X}}$  defined by

$$\mathcal{C}_{\mathcal{X}} = \{\alpha = [0; a_1, a_2, \dots] \in (0, 1) : \{a_n\}_{n \geq 1} \in \mathcal{X}\}.$$

In particular, if a real number  $\alpha$  lies in  $\mathcal{C}_{\mathcal{X}}$ , then this is also the case for every  $\beta$  in  $\mathcal{C}_{\alpha} := \overline{\{T^n(\alpha)\}_{n \geq 0}}$ , where  $T$  denotes the Gauss map, defined from  $(0, 1)$  into itself by  $T(x) = \{1/x\}$ . Indeed, we clearly have  $T([0; a_1, a_2, \dots]) = [0; a_2, a_3, \dots]$ . A way to investigate the question of the complexity of the continued fraction expansion of  $\alpha$  is to determine the behaviour of the sequence  $\{T^n(\alpha)\}_{n \geq 0}$  or, equivalently, to determine the structure of the underlying dynamical system  $(\mathcal{C}_{\alpha}, T)$ . Roughly speaking, we can consider that the larger  $\mathcal{C}_{\alpha}$  is, the more complex is the continued fraction expansion of  $\alpha$ .

Thus, if the symbolic dynamical system  $\mathcal{X}$  has a too simple structure, for instance if it has low complexity, then we can expect that no algebraic number of degree at least three lies in the set  $\mathcal{C}_{\mathcal{X}}$ .

*Problem 3.* Let  $\mathcal{X}$  be a subshift on  $\mathbf{Z}_{\geq 1}$  with sublinear complexity, that is, whose complexity function satisfies  $p_{\mathcal{X}}(n) \leq Mn$  for some absolute constant  $M$  and every positive integer  $n$ . Does the set  $\mathcal{C}_{\mathcal{X}}$  only contain quadratic or transcendental numbers?

Only very partial results are known in the direction of Problem 3. A famous result of Morse and Hedlund [19] states that a subshift  $\mathcal{X}$  whose complexity function satisfies  $p_{\mathcal{X}}(n) \leq n$  for some positive integer  $n$  must be periodic. In that case, it follows that  $\mathcal{C}_{\mathcal{X}}$  is a finite set composed only of quadratic numbers. Further, it is shown in [5] that for a Sturmian subshift  $\mathcal{X}$ , that is, a subshift with complexity  $p_{\mathcal{X}}(n) = n + 1$  for every  $n \geq 1$ , the set  $\mathcal{C}_{\mathcal{X}}$  is an uncountable set composed only of transcendental numbers. Theorem 4 slightly improves this result.

**THEOREM 4.** *Let  $\mathcal{X}$  be a subshift on  $\mathbf{Z}_{\geq 1}$ . If the set  $\mathcal{C}_{\mathcal{X}}$  contains a real algebraic number of degree at least three, then the complexity function of  $\mathcal{X}$  satisfies*

$$\lim_{n \rightarrow +\infty} p_{\mathcal{X}}(n) - n = +\infty.$$

Linearly recurrent subshifts form a class of particular interest of subshifts of low complexity. Let  $\mathcal{X}=(X, S)$  be a subshift and  $W$  be a finite word. The cylinder associated with  $W$  is, by definition, the subset  $\langle W \rangle$  of  $X$  formed by the sequences that begin in the word  $W$ . A minimal subshift  $(X, S)$  is linearly recurrent if there exists a positive constant  $c$  such that for each cylinder  $\langle W \rangle$  the return time to  $\langle W \rangle$  under  $S$  is bounded by  $c|W|$ . Such dynamical systems, studied, e.g., in [14], are uniquely ergodic and have low complexity (in particular, they have zero entropy), but without being necessarily trivial. Another contribution to Problem 3 is given by Theorem 5.

**THEOREM 5.** *Let  $\mathcal{X}$  be a linearly recurrent subshift on  $\mathbf{Z}_{\geq 1}$ . Then the set  $\mathcal{C}_{\mathcal{X}}$  is composed only by quadratic or transcendental numbers.*

The proofs of Theorems 3–5 are postponed to §5.

#### 4. Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 rest on the following deep result, commonly known as the Schmidt subspace theorem.

**THEOREM A** (W.M. Schmidt). *Let  $m \geq 2$  be an integer. Let  $L_1, \dots, L_m$  be linearly independent linear forms in  $\mathbf{x}=(x_1, \dots, x_m)$  with algebraic coefficients. Let  $\varepsilon$  be a positive real number. Then the set of solutions  $\mathbf{x}=(x_1, \dots, x_m)$  in  $\mathbf{Z}^m$  to the inequality*

$$|L_1(\mathbf{x}) \dots L_m(\mathbf{x})| \leq (\max\{|x_1|, \dots, |x_m|\})^{-\varepsilon}$$

*lies in finitely many proper subspaces of  $\mathbf{Q}^m$ .*

*Proof.* See, e.g., [25] or [26]. The case  $m=3$  has been established earlier in [24].  $\square$

Compared with the pioneering work [12] and the recent papers [21], [5], [13] and [8], the novelty in the present paper is that we are able to use Theorem A with  $m=4$  and not only with  $m=3$ , as in all of these works.

We further need an easy auxiliary result.



LEMMA 1. *Let  $\alpha=[a_0; a_1, a_2, \dots]$  and  $\beta=[b_0; b_1, b_2, \dots]$  be real numbers. Assume that, for some positive integer  $m$ , we have  $a_j=b_j$  for  $j=0, \dots, m$ . Then we have*

$$|\alpha - \beta| < q_m^{-2},$$

where  $q_m$  is the denominator of the convergent  $[a_0; a_1, \dots, a_m]$ .

*Proof.* Since  $[a_0; a_1, \dots, a_m]=:p_m/q_m$  is a convergent to  $\alpha$  and to  $\beta$ , the real numbers  $\alpha - p_m/q_m$  and  $\beta - p_m/q_m$  have the same sign and are both in absolute value less than  $q_m^{-2}$ , and hence the lemma follows.  $\square$

Now, we have all the tools to establish Theorems 1 and 2.

*Proof of Theorem 1.* Keep the notation and the hypothesis of this theorem. Assume that the parameter  $w > 1$  is fixed, as well as the sequence  $\{V_n\}_{n \geq 1}$  occurring in the definition of Condition  $(*)_w$ . Set also  $s_n = |V_n|$  for  $n \geq 1$ . We want to prove that the real number

$$\alpha := [0; a_1, a_2, \dots]$$

is transcendental. We assume that  $\alpha$  is algebraic of degree at least three, and we aim at deriving a contradiction. Throughout this section, the constants implied by  $\ll$  depend only on  $\alpha$ .

Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to  $\alpha$ . Observe first that we have

$$q_{l+1} \ll q_l^{1.1} \quad \text{for } l \geq 1, \tag{2}$$

by Roth's theorem [23].

The key fact for the proof of Theorem 1 is the observation that  $\alpha$  admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for positive integers  $n$ , we define the sequence  $\{b_k^{(n)}\}_{k \geq 1}$  by

$$b_{h+j s_n}^{(n)} = a_h \quad \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0.$$

The sequence  $\{b_k^{(n)}\}_{k \geq 1}$  is purely periodic with period  $V_n$ . Set

$$\alpha_n = [0; b_1^{(n)}, b_2^{(n)}, \dots]$$

and observe that  $\alpha_n$  is a root of the quadratic polynomial

$$P_n(X) := q_{s_n-1} X^2 + (q_{s_n} - p_{s_n-1}) X - p_{s_n}.$$

By Rolle's theorem and Lemma 1, for positive integers  $n$ , we have

$$|P_n(\alpha)| = |P_n(\alpha) - P_n(\alpha_n)| \ll q_{s_n} |\alpha - \alpha_n| \ll q_{s_n} q_{[ws_n]}^{-2}, \quad (3)$$

since the first  $[ws_n]$  partial quotients of  $\alpha$  and  $\alpha_n$  are the same. Furthermore, we clearly have

$$|q_{s_n} \alpha - p_{s_n}| \leq q_{s_n}^{-1}, \quad (4)$$

and we infer from (2) that

$$|q_{s_n-1} \alpha - p_{s_n-1}| \leq q_{s_n-1}^{-1} \ll q_{s_n}^{-0.9}. \quad (5)$$

Consider now the four linearly independent linear forms

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) &= \alpha^2 X_2 + \alpha(X_1 - X_4) - X_3, \\ L_2(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3, \\ L_3(X_1, X_2, X_3, X_4) &= X_1, \\ L_4(X_1, X_2, X_3, X_4) &= X_2. \end{aligned}$$

Evaluating them on the quadruple  $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$ , it follows from (3) and (4) that

$$\prod_{1 \leq j \leq 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| \ll q_{s_n}^2 q_{[ws_n]}^{-2}. \quad (6)$$

By assumption, there exists a real number  $M$  such that  $\log q_l \leq l \log M$  for positive integers  $l$ . Furthermore, an immediate induction shows that  $q_{l+2} \geq 2q_l$  holds for positive integers  $l$ . Consequently, for integers  $n \geq 3$ , we get

$$\frac{q_{[ws_n]}}{q_{s_n}} \geq \sqrt{2}^{[(w-1)s_n]-1} \geq q_{s_n}^{(w-1-2/s_n)(\log \sqrt{2})/\log M},$$

and we infer from (6) and  $w > 1$  that

$$\prod_{1 \leq j \leq 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| \ll q_{s_n}^{-\varepsilon}$$

holds for some positive real number  $\varepsilon$ , when  $n$  is large enough.

It then follows from Theorem A that the points  $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$  lie in a finite number of proper subspaces of  $\mathbf{Q}^4$ . Thus, there exist a non-zero integer quadruple  $(x_1, x_2, x_3, x_4)$  and an infinite set of distinct positive integers  $\mathcal{N}_1$  such that

$$x_1 q_{s_n} + x_2 q_{s_n-1} + x_3 p_{s_n} + x_4 p_{s_n-1} = 0 \quad (7)$$

for  $n$  in  $\mathcal{N}_1$ . Observe that  $(x_2, x_4) \neq (0, 0)$ , since, otherwise, by letting  $n$  tend to infinity along  $\mathcal{N}_1$  in (7), we would get that the real number  $\alpha$  is rational. Dividing (7) by  $q_{s_n}$ , we obtain

$$x_1 + x_2 \frac{q_{s_n-1}}{q_{s_n}} + x_3 \frac{p_{s_n}}{q_{s_n}} + x_4 \frac{p_{s_n-1}}{q_{s_n-1}} \frac{q_{s_n-1}}{q_{s_n}} = 0. \quad (8)$$

By letting  $n$  tend to infinity along  $\mathcal{N}_1$  in (8), we get that

$$\beta := \lim_{\mathcal{N}_1 \ni n \rightarrow +\infty} \frac{q_{s_n-1}}{q_{s_n}} = -\frac{x_1 + x_3 \alpha}{x_2 + x_4 \alpha}.$$

Furthermore, observe that, for  $n$  in  $\mathcal{N}_1$ , we have

$$\left| \beta - \frac{q_{s_n-1}}{q_{s_n}} \right| = \left| \frac{x_1 + x_3 \alpha}{x_2 + x_4 \alpha} - \frac{x_1 + x_3 p_{s_n}/q_{s_n}}{x_2 + x_4 p_{s_n-1}/q_{s_n-1}} \right| \ll \frac{1}{q_{s_n-1}^2} \ll \frac{1}{q_{s_n}^{1.8}}, \quad (9)$$

by (4) and (5). Since  $q_{s_n-1}$  and  $q_{s_n}$  are coprime and  $s_n$  tends to infinity when  $n$  tends to infinity along  $\mathcal{N}_1$ , this implies that  $\beta$  is irrational.

Consider now the three linearly independent linear forms

$$L'_1(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \quad L'_2(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \quad L'_3(Y_1, Y_2, Y_3) = Y_1.$$

Evaluating them on the triple  $(q_{s_n}, q_{s_n-1}, p_{s_n})$  with  $n \in \mathcal{N}_1$ , we infer from (4) and (9) that

$$\prod_{1 \leq j \leq 3} |L'_j(q_{s_n}, q_{s_n-1}, p_{s_n})| \ll q_{s_n}^{-0.8}.$$

It then follows from Theorem A that the points  $(q_{s_n}, q_{s_n-1}, p_{s_n})$  with  $n \in \mathcal{N}_1$  lie in a finite number of proper subspaces of  $\mathbf{Q}^3$ . Thus, there exist a non-zero integer triple  $(y_1, y_2, y_3)$  and an infinite set of distinct positive integers  $\mathcal{N}_2$  such that

$$y_1 q_{s_n} + y_2 q_{s_n-1} + y_3 p_{s_n} = 0 \quad (10)$$

for  $n$  in  $\mathcal{N}_2$ . Dividing (10) by  $q_{s_n}$  and letting  $n$  tend to infinity along  $\mathcal{N}_2$ , we get

$$y_1 + y_2 \beta + y_3 \alpha = 0. \quad (11)$$

To obtain another equation linking  $\alpha$  and  $\beta$ , we consider the three linearly independent linear forms

$$L''_1(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \quad L''_2(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \quad L''_3(Z_1, Z_2, Z_3) = Z_1.$$

Evaluating them on the triple  $(q_{s_n}, q_{s_n-1}, p_{s_n-1})$  with  $n \in \mathcal{N}_1$ , we infer from (5) and (9) that

$$\prod_{1 \leq j \leq 3} |L''_j(q_{s_n}, q_{s_n-1}, p_{s_n-1})| \ll q_{s_n}^{-0.7}.$$

It then follows from Theorem A that the points  $(q_{s_n}, q_{s_n-1}, p_{s_n-1})$  with  $n \in \mathcal{N}_1$  lie in a finite number of proper subspaces of  $\mathbf{Q}^3$ . Thus, there exist a non-zero integer triple  $(z_1, z_2, z_3)$  and an infinite set of distinct positive integers  $\mathcal{N}_3$  such that

$$z_1 q_{s_n} + z_2 q_{s_n-1} + z_3 p_{s_n-1} = 0 \quad (12)$$

for  $n$  in  $\mathcal{N}_3$ . Dividing (12) by  $q_{s_n-1}$  and letting  $n$  tend to infinity along  $\mathcal{N}_3$ , we get

$$\frac{z_1}{\beta} + z_2 + z_3 \alpha = 0. \quad (13)$$

Observe that  $y_2 \neq 0$  since  $\alpha$  is irrational. We infer from (11) and (13) that

$$(z_3 \alpha + z_2)(y_3 \alpha + y_1) = y_2 z_1.$$

If  $y_3 z_3 = 0$ , then (11) and (13) yield that  $\beta$  is rational, which is a contradiction. Consequently,  $y_3 z_3 \neq 0$  and  $\alpha$  is a quadratic real number, which is again a contradiction. This completes the proof of the second assertion of the theorem.

It then remains for us to explain why we can drop the assumption on the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  when  $w$  is sufficiently large. We return to the beginning of the proof, and we assume that  $w \geq 2$ . Using well-known facts from the theory of continuants (see, e.g., [20]), inequality (3) becomes

$$|P_n(\alpha)| \ll q_{s_n} q_{2s_n}^{-2} \ll q_{s_n} q_{s_n}^{-4} \ll q_{s_n}^{-3} \ll H(P_n)^{-3},$$

where  $H(P_n)$  denotes the height of the polynomial  $P_n$ , that is, the maximum of the absolute values of its coefficients. By the main result from [24] (or by using Theorem A with  $m=3$  and the linear forms  $\alpha^2 X_2 + \alpha X_1 + X_0$ ,  $X_2$  and  $X_1$ ), this immediately implies that  $\alpha$  is transcendental.  $\square$

*Proof of Theorem 2.* Assume that the parameters  $w$  and  $w'$  are fixed, as well as the sequences  $\{U_n\}_{n \geq 1}$  and  $\{V_n\}_{n \geq 1}$  occurring in the definition of Condition  $(**)_{w,w'}$ . Without any loss of generality, we add in the statement of Condition  $(**)_{w,w'}$  the following two assumptions:

- (iv) The sequence  $\{|U_n|\}_{n \geq 1}$  is unbounded;
- (v) For  $n \geq 1$ , the last letter of the word  $U_n$  differs from the last letter of the word  $V_n$ .

We point out that the conditions (iv) and (v) do not at all restrict the generality. Indeed, if (iv) is not fulfilled by a sequence  $\mathbf{a}$  satisfying (i)–(iii) of Condition  $(**)_{w,w'}$ , then the desired result follows from Theorem 1. To see that (v) does not cause any trouble, we make the following observation. Let  $a$  be a letter, and  $U$  and  $V$  be two words

such that  $\mathbf{a}$  begins with  $Ua(Va)^w$ . Then  $\mathbf{a}$  also begins with  $U(aV)^w$ , and we have trivially  $|U|/|aV| \leq |Ua|/|Va|$ .

Set  $r_n = |U_n|$  and  $s_n = |V_n|$  for  $n \geq 1$ . We want to prove that the real number

$$\alpha := [0; a_1, a_2, \dots]$$

is transcendental. We assume that  $\alpha$  is algebraic of degree at least three, and we aim at deriving a contradiction. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to  $\alpha$ .

Let  $n$  be a positive integer. Since  $w > 1$  and  $r_n \leq w's_n$ , we get

$$\frac{2r_n + s_n}{r_n + ws_n} \leq \frac{2w's_n + s_n}{w's_n + ws_n} = \frac{2w' + 1}{w' + w} < \frac{\log m}{\log M},$$

by (1). Consequently, there exist positive real numbers  $\eta$  and  $\eta'$  with  $\eta < 1$  such that

$$(1 + \eta)(1 + \eta')(2r_n + s_n) \log M < (1 - \eta')(r_n + ws_n) \log m \tag{14}$$

for  $n \geq 1$ . Notice that we have

$$q_{l+1} \ll q_l^{1+\eta} \quad \text{for } l \geq 1, \tag{15}$$

by Roth's theorem [23].

As for the proof of Theorem 1, we observe that  $\alpha$  admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for positive integers  $n$ , we define the sequence  $\{b_k^{(n)}\}_{k \geq 1}$  by

$$\begin{aligned} b_h^{(n)} &= a_h && \text{for } 1 \leq h \leq r_n + s_n, \\ b_{r_n+h+j s_n}^{(n)} &= a_{r_n+h} && \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0. \end{aligned}$$

The sequence  $\{b_k^{(n)}\}_{k \geq 1}$  is eventually periodic, with preperiod  $U_n$  and with period  $V_n$ . Set

$$\alpha_n = [0; b_1^{(n)}, b_2^{(n)}, \dots]$$

and observe that  $\alpha_n$  is a root of the quadratic polynomial

$$\begin{aligned} P_n(X) := & (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ & - (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} + p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1})X \\ & + (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

For positive integers  $n$ , we infer from Rolle's theorem and Lemma 1 that

$$|P_n(\alpha)| = |P_n(\alpha) - P_n(\alpha_n)| \ll q_{r_n} q_{r_n+s_n} |\alpha - \alpha_n| \ll q_{r_n} q_{r_n+s_n} q_{r_n+[ws_n]}^{-2}, \quad (16)$$

since the first  $r_n + [ws_n]$  partial quotients of  $\alpha$  and  $\alpha_n$  are the same. Furthermore, by (15), we have

$$|(q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1})\alpha - (q_{r_n-1} p_{r_n+s_n} - q_{r_n} p_{r_n+s_n-1})| \ll q_{r_n} q_{r_n+s_n}^{-1+\eta} \quad (17)$$

and

$$|(q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1})\alpha - (p_{r_n-1} q_{r_n+s_n} - p_{r_n} q_{r_n+s_n-1})| \ll q_{r_n}^{-1+\eta} q_{r_n+s_n}. \quad (18)$$

We have as well the obvious upper bound

$$|q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1}| \leq q_{r_n} q_{r_n+s_n}. \quad (19)$$

Consider now the four linearly independent linear forms

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) &= \alpha^2 X_1 - \alpha(X_2 + X_3) + X_4, \\ L_2(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2, \\ L_3(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3, \\ L_4(X_1, X_2, X_3, X_4) &= X_1. \end{aligned}$$

Evaluating them on the quadruple

$$\begin{aligned} \tilde{z}_n := & (q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1}, q_{r_n-1} p_{r_n+s_n} - q_{r_n} p_{r_n+s_n-1}, \\ & p_{r_n-1} q_{r_n+s_n} - p_{r_n} q_{r_n+s_n-1}, p_{r_n-1} p_{r_n+s_n} - p_{r_n} p_{r_n+s_n-1}), \end{aligned}$$

it follows from (16), (17), (18) and (19) that

$$\prod_{1 \leq j \leq 4} |L_j(\tilde{z}_n)| \ll q_{r_n}^{2+\eta} q_{r_n+s_n}^{2+\eta} q_{r_n+[ws_n]}^{-2} \ll (q_{r_n} q_{r_n+s_n})^{-\eta} (q_{r_n}^{1+\eta} q_{r_n+s_n}^{1+\eta} q_{r_n+[ws_n]}^{-1})^2.$$

Assuming  $n$  sufficiently large, we have

$$q_{r_n} \leq M^{(1+\eta')r_n}, \quad q_{r_n+s_n} \leq M^{(1+\eta')(r_n+s_n)} \quad \text{and} \quad q_{r_n+[ws_n]} \geq m^{(1-\eta')(r_n+ws_n)},$$

with  $\eta'$  as in (14). Consequently, we get

$$(q_{r_n}^{1+\eta} q_{r_n+s_n}^{1+\eta} q_{r_n+[ws_n]}^{-1}) \leq M^{(1+\eta)(1+\eta')(2r_n+s_n)} m^{-(1-\eta')(r_n+ws_n)} \leq 1,$$

by (14). Thus, we get the upper bound

$$\prod_{1 \leq j \leq 4} |L_j(\underline{z}_n)| \ll (q_{r_n} q_{r_n+s_n})^{-\eta}$$

for positive integers  $n$ .

It then follows from Theorem A that the points  $\underline{z}_n$  lie in a finite number of proper subspaces of  $\mathbf{Q}^4$ . Thus, there exist a non-zero integer quadruple  $(x_1, x_2, x_3, x_4)$  and an infinite set of distinct positive integers  $\mathcal{N}_1$  such that

$$\begin{aligned} x_1(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}) &+ x_2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}) \\ &+ x_3(p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}) + x_4(p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}) = 0 \end{aligned} \quad (20)$$

for  $n$  in  $\mathcal{N}_1$ .

Divide (20) by  $q_{r_n}q_{r_n+s_n-1}$  and observe that  $p_{r_n}/q_{r_n}$  and  $p_{r_n+s_n}/q_{r_n+s_n}$  tend to  $\alpha$  as  $n$  tends to infinity along  $\mathcal{N}_1$ . Taking the limit, we get that either

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0 \quad (21)$$

or

$$\frac{q_{r_n-1}q_{r_n+s_n}}{q_{r_n}q_{r_n+s_n-1}} \rightarrow 1, \quad \text{as } n \rightarrow \infty \text{ along } \mathcal{N}_1, \quad (22)$$

must hold. In the former case, since  $\alpha$  is irrational and not quadratic, we get that  $x_1 = x_4 = 0$  and  $x_2 = -x_3$ . Then  $x_2$  is non-zero and, for  $n$  in  $\mathcal{N}_1$ , we have

$$q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} = p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}.$$

Thus, the polynomial  $P_n(X)$  can simply be expressed as

$$\begin{aligned} P_n(X) &:= (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ &\quad - 2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1})X + (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

Consider now the three linearly independent linear forms

$$\begin{aligned} L'_1(Y_1, Y_2, Y_3) &= \alpha^2 X_1 - 2\alpha X_2 + X_3, \\ L'_2(Y_1, Y_2, Y_3) &= \alpha X_1 - X_2, \\ L'_3(Y_1, Y_2, Y_3) &= X_1. \end{aligned}$$

Evaluating them on the triple

$$\begin{aligned} \underline{z}'_n &:= (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}, q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}, \\ &\quad p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}), \end{aligned}$$

it follows from (16), (17) and (19) that

$$\prod_{1 \leq j \leq 3} |L'_j(\underline{z}'_n)| \ll q_{r_n}^3 q_{r_n+s_n}^{1+\eta} q_{r_n+[ws_n]}^{-2} \ll q_{r_n}^2 q_{r_n+s_n}^{2+\eta} q_{r_n+[ws_n]}^{-2} \ll (q_{r_n} q_{r_n+s_n})^{-\eta},$$

by the above computation.

It then follows from Theorem A that the points  $\underline{z}'_n$  lie in a finite number of proper subspaces of  $\mathbf{Q}^3$ . Thus, there exist a non-zero integer triple  $(x'_1, x'_2, x'_3)$  and an infinite set of distinct positive integers  $\mathcal{N}'_2$  included in  $\mathcal{N}_1$  such that

$$\begin{aligned} x'_1(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}) + x'_2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}) \\ + x'_3(p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}) = 0 \end{aligned} \quad (23)$$

for  $n$  in  $\mathcal{N}'_2$ .

Divide (23) by  $q_{r_n}q_{r_n+s_n-1}$  and observe that  $p_{r_n}/q_{r_n}$  and  $p_{r_n+s_n}/q_{r_n+s_n}$  tend to  $\alpha$  as  $n$  tends to infinity along  $\mathcal{N}'_2$ . Taking the limit, we get that either

$$x'_1 + x'_2\alpha + x'_3\alpha^2 = 0 \quad (24)$$

or

$$\frac{q_{r_n-1}q_{r_n+s_n}}{q_{r_n}q_{r_n+s_n-1}} \rightarrow 1, \quad \text{as } n \rightarrow \infty \text{ along } \mathcal{N}'_2, \quad (25)$$

must hold. In the former case, we have a contradiction since  $\alpha$  is irrational and not quadratic.

Consequently, to conclude the proof of our theorem, it is enough to derive a contradiction from (22) (resp. from (25)), assuming that (21) (resp. (24)) does not hold. To this end, we observe that (20) (resp. (23)) allows us to control the speed of convergence of  $Q_n := (q_{r_n-1}q_{r_n+s_n})/(q_{r_n}q_{r_n+s_n-1})$  to 1 along  $\mathcal{N}'_1$  (resp. along  $\mathcal{N}'_2$ ).

Thus, we assume that the quadruple  $(x_1, x_2, x_3, x_4)$  obtained after the first application of Theorem A satisfies  $x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 \neq 0$ . Dividing (20) by  $q_{r_n}q_{r_n+s_n-1}$ , we get

$$\begin{aligned} x_1(Q_n - 1) + x_2 \left( Q_n \frac{p_{r_n+s_n}}{q_{r_n+s_n}} - \frac{p_{r_n+s_n-1}}{q_{r_n+s_n-1}} \right) + x_3 \left( Q_n \frac{p_{r_n-1}}{q_{r_n-1}} - \frac{p_{r_n}}{q_{r_n}} \right) \\ + x_4 \left( Q_n \frac{p_{r_n-1}}{q_{r_n-1}} \frac{p_{r_n+s_n}}{q_{r_n+s_n}} - \frac{p_{r_n}}{q_{r_n}} \frac{p_{r_n+s_n-1}}{q_{r_n+s_n-1}} \right) = 0 \end{aligned} \quad (26)$$

for  $n$  in  $\mathcal{N}'_1$ . To shorten the notation, for  $l \geq 1$ , we put  $R_l := \alpha - p_l/q_l$ . We rewrite (26) as

$$\begin{aligned} x_1(Q_n - 1) + x_2(Q_n(\alpha - R_{r_n+s_n}) - (\alpha - R_{r_n+s_n-1})) \\ + x_3(Q_n(\alpha - R_{r_n-1}) - (\alpha - R_{r_n})) \\ + x_4(Q_n(\alpha - R_{r_n-1})(\alpha - R_{r_n+s_n}) - (\alpha - R_{r_n})(\alpha - R_{r_n+s_n-1})) = 0. \end{aligned}$$



This yields

$$\begin{aligned}
 & (Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2) \\
 &= x_2 Q_n R_{r_n + s_n} - x_2 R_{r_n + s_n - 1} + x_3 Q_n R_{r_n - 1} - x_3 R_{r_n} \\
 &\quad - x_4 Q_n R_{r_n - 1} R_{r_n + s_n} + x_4 R_{r_n} R_{r_n + s_n - 1} \\
 &\quad + \alpha(x_4 Q_n R_{r_n - 1} + x_4 Q_n R_{r_n + s_n} - x_4 R_{r_n} - x_4 R_{r_n + s_n - 1}).
 \end{aligned} \tag{27}$$

Observe that  $|R_l| \leq 1/q_l^2$  for  $l \geq 1$ . Furthermore, for  $n$  large enough, we have  $\frac{1}{2} \leq Q_n \leq 2$ , by our assumption (22). Consequently, we derive from (27) that

$$|(Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2)| \ll |R_{r_n - 1}| \ll q_{r_n - 1}^{-2}.$$

Since we have assumed that (21) does not hold, we get

$$|Q_n - 1| \ll q_{r_n - 1}^{-2}. \tag{28}$$

On the other hand, observe that the rational number  $Q_n$  is the quotient of the two continued fractions  $[a_{r_n + s_n}; a_{r_n + s_n - 1}, \dots, a_1]$  and  $[a_{r_n}; a_{r_n - 1}, \dots, a_1]$ . By assumption (v) from Condition (\*\*)<sub>w, w'</sub>, we have  $a_{r_n + s_n} \neq a_{r_n}$ , and thus either  $a_{r_n + s_n} - a_{r_n} \geq 1$  or  $a_{r_n} - a_{r_n + s_n} \geq 1$  holds. A simple calculation then shows that

$$|Q_n - 1| \gg a_{r_n}^{-1} \min\{a_{r_n + s_n - 1}^{-1} + a_{r_n - 2}^{-1}, a_{r_n + s_n - 2}^{-1} + a_{r_n - 1}^{-1}\} \gg a_{r_n}^{-1} q_{r_n - 1}^{-1},$$

since  $q_{r_n - 1} \geq \max\{a_{r_n - 1}, a_{r_n - 2}\}$ . Combined with (28), this gives  $a_{r_n} \gg q_{r_n - 1}$  and

$$q_{r_n} \geq a_{r_n} q_{r_n - 1} \gg q_{r_n - 1}^2. \tag{29}$$

Since  $\eta < 1$  and (29) holds for infinitely many  $n$ , we get a contradiction with (15).

We derive a contradiction from (25) in an entirely similar way. This completes the proof of our theorem.  $\square$

## 5. Proofs of Theorems 3–5

Before establishing Theorems 3–5, we state an easy, but useful, auxiliary result.

**LEMMA 2.** *Let  $\sigma$  be a prolongable morphism defined on a finite alphabet  $\mathcal{A}$ . Let  $\mathbf{a}$  be the associated fixed point and  $a$  be the first letter of  $\mathbf{a}$ . Then there exists a positive constant  $c$  such that, for every positive integer  $n$  and every letter  $b$  occurring in  $\mathbf{a}$ , we have  $|\sigma^n(a)| \geq c|\sigma^n(b)|$ .*

*Proof.* Without loss of generality, we may assume that  $\mathcal{A}$  is exactly the set of letters occurring in  $\mathbf{a}$ . Let  $b$  be in  $\mathcal{A}$ . Since  $\mathbf{a}$  is obtained as the limit  $\lim_{n \rightarrow +\infty} \sigma^n(a)$ , there exists an integer  $n_b$  such that the word  $\sigma^{n_b}(a)$  contains the letter  $b$ . Set

$$s = \max_{b \in \mathcal{A}} \{|\sigma(b)|\} \quad \text{and} \quad n_0 = \max_{b \in \mathcal{A}} \{n_b\}.$$

Let  $n$  be a positive integer. If  $n \leq n_0$ , then we have

$$|\sigma^n(a)| \geq 1 \geq \frac{1}{s^{n_0}} |\sigma^n(b)|.$$

If  $n > n_0$ , then we get

$$|\sigma^n(a)| = |\sigma^{n-n_0}(\sigma^{n_0}(a))| \geq |\sigma^{n-n_0}(b)| \geq \frac{1}{s^{n_0}} |\sigma^n(b)|,$$

and the lemma follows by taking  $c = s^{-n_0}$ .  $\square$

*Proof of Theorem 3.* Let us assume that  $\mathbf{b}$  is a sequence generated by a recurrent morphism  $\sigma$  and that  $\mathbf{b}$  is not eventually periodic. There exists a fixed point  $\mathbf{a}$  of  $\sigma$  and a coding  $\varphi$  such that  $\mathbf{b} = \varphi(\mathbf{a})$ . By assumption, the first letter  $a$  which occurs in  $\mathbf{a}$  should appear at least twice. Thus, there exists a finite (possibly empty) word  $W$  such that  $aWa$  is a prefix of the word  $\mathbf{a}$ . We check that the assumptions of Theorem 1 are satisfied by  $\mathbf{b}$  with the sequence  $\{V_n\}_{n \geq 1}$  defined by  $V_n = \varphi(\sigma^n(aW))$  for  $n \geq 1$ . Indeed, by Lemma 2, there exists a positive rational number  $c$ , depending only on  $\sigma$  and  $W$ , such that

$$|\sigma^n(a)| \geq c |\sigma^n(aW)|.$$

This implies that  $\varphi(\sigma^n(aWa))$  begins in  $(\varphi(\sigma^n(aW)))^{1+c}$ . Since  $\varphi(\sigma^n(aWa))$  is a prefix of  $\mathbf{a}$ , we get that the sequence  $\mathbf{b}$  satisfies Condition  $(*)_{1+c}$ . We conclude by applying Theorem 1.  $\square$

*Proof of Theorem 4.* Let  $\mathcal{X} = (X, S)$  be a subshift such that  $p_{\mathcal{X}}(n) - n$  is bounded and let  $\alpha$  be an element of  $\mathcal{C}_{\mathcal{X}}$ . By definition of the set  $\mathcal{C}_{\mathcal{X}}$ , there exists a sequence  $\mathbf{a} = \{a_l\}_{l \geq 1}$  in  $X$  such that  $\alpha = [0; a_1, a_2, \dots]$ .

First, assume that the complexity function of the sequence  $\mathbf{a}$  satisfies  $p_{\mathbf{a}}(n) \leq n$  for some  $n$ . It follows from a theorem of Morse and Hedlund [19] that  $\mathbf{a}$  is eventually periodic, and thus  $\alpha$  is a quadratic number.

Now, assume that  $p_{\mathcal{X}}(n) > n$  for every integer  $n$ . Since  $p_{\mathcal{X}}(n) - n$  is bounded, there exist two positive integers  $n_0$  and  $a$  such that  $p_{\mathcal{X}}(n) = n + a$  for all  $n \geq n_0$  (see, e.g., [4]). This implies (see, e.g., [10]) that there exist a finite word  $W$ , a non-erasing morphism  $\phi$  and a Sturmian sequence  $\mathbf{u}$  such that  $\mathbf{a} = W\phi(\mathbf{u})$ . Since  $\mathbf{u}$  begins with arbitrarily long squares (this is proved in [5]) and since  $\phi$  is a non-erasing morphism, it follows that  $\phi(\mathbf{u})$  also begins with arbitrarily long squares, and hence it satisfies Condition  $(*)_2$ . We then infer from Theorem 1 that the real number  $\alpha' = [0; a_{|W|+1}, a_{|W|+2}, a_{|W|+3}, \dots]$  is transcendental. It immediately follows that  $\alpha$  is a transcendental number, concluding the proof of the theorem.  $\square$

*Proof of Theorem 5.* Let  $\mathcal{X}=(X, S)$  be a linearly recurrent subshift and let  $\alpha$  be an element of  $\mathcal{C}_{\mathcal{X}}$ . By the definition of the set  $\mathcal{C}_{\mathcal{X}}$ , there exists a sequence  $\mathbf{a}=\{a_l\}_{l \geq 1}$  in  $X$  such that  $\alpha=[0; a_1, a_2, \dots]$ . By assumption, there exists a positive integer  $k$  such that the gap between two consecutive occurrences in  $\mathbf{a}$  of every factor  $W$  of length  $n$  is at most  $kn$ . For every positive integer  $n$ , let  $U_n$  denote the prefix of length  $n$  of  $\mathbf{a}$ , and let  $W_n$  be the word of length  $kn$  defined by  $U_{(k+1)n}=U_nW_n$ . Since, by assumption,  $U_n$  has at least one occurrence in the word  $W_n$ , there exist two (possibly empty) finite words  $A_n$  and  $B_n$  such that  $W_n=A_nU_nB_n$ . It follows that  $U_nA_nU_n$  is a prefix of  $\mathbf{a}$  and, moreover,  $U_nA_nU_n=(U_nA_n)^w$  for some rational number  $w$  with  $w \geq 1+1/k$ . Then either  $\mathbf{a}$  is eventually periodic (in which case,  $\alpha$  is a quadratic number) or  $\mathbf{a}$  satisfies Condition  $(*)_{1+1/k}$ , and the transcendence of  $\alpha$  follows from Theorem 1, concluding the proof.  $\square$

*Acknowledgement.* We would like to warmly thank Jean-Paul Allouche for many useful remarks. The first author is also most grateful to Valérie Berthé for her constant help and support.

### References

- [1] ADAMCZEWSKI, B. & BUGEAUD, Y., On the complexity of algebraic numbers I. Expansions in integer bases. To appear in *Ann. of Math.*
- [2] ADAMCZEWSKI, B., BUGEAUD, Y. & DAVISON, J. L., Continued fractions and transcendental numbers. Preprint, 2005.
- [3] ADAMCZEWSKI, B., BUGEAUD, Y. & LUCA, F., Sur la complexité des nombres algébriques. *C. R. Acad. Sci. Paris*, 339 (2004), 11–14.
- [4] ALLOUCHE, J.-P., Nouveaux résultats de transcendance de réels à développement non aléatoire. *Gaz. Math.*, 84 (2000), 19–34.
- [5] ALLOUCHE, J.-P., DAVISON, J. L., QUEFFÉLEC, M. & ZAMBONI, L. Q., Transcendence of Sturmian or morphic continued fractions. *J. Number Theory*, 91 (2001), 39–66.
- [6] ALLOUCHE, J.-P. & SHALLIT, J., *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge Univ. Press, Cambridge, 2003.
- [7] BAILEY, D. H., BORWEIN, J. M., CRANDALL, R. E. & POMERANCE, C., On the binary expansions of algebraic numbers. *J. Théor. Nombres Bordeaux*, 16 (2004), 487–518.
- [8] BAXA, C., Extremal values of continuants and transcendence of certain continued fractions. *Adv. in Appl. Math.*, 32 (2004), 754–790.
- [9] BOREL, É., Sur les chiffres décimaux de  $\sqrt{2}$  et divers problèmes de probabilités en chaîne. *C. R. Acad. Sci. Paris*, 230 (1950), 591–593.
- [10] CASSAIGNE, J., Sequences with grouped factors, in *Developments in Language Theory*, III (Thessaloniki, 1997), pp. 211–222. World Sci. Publishing, River Edge, NJ, 1998.
- [11] COBHAM, A., Uniform tag sequences. *Math. Systems Theory*, 6 (1972), 164–192.
- [12] DAVISON, J. L., A class of transcendental numbers with bounded partial quotients, in *Number Theory and Applications* (Banff, AB, 1988), pp. 365–371. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 265. Kluwer, Dordrecht, 1989.

- [13] — Continued fractions with bounded partial quotients. *Proc. Edinburgh Math. Soc.*, 45 (2002), 653–671.
- [14] DURAND, F., Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergodic Theory Dynam. Systems*, 20 (2000), 1061–1078; Corrigendum and addendum. *Ibid.*, 23 (2003), 663–669.
- [15] HARTMANIS, J. & STEARNS, R. E., On the computational complexity of algorithms. *Trans. Amer. Math. Soc.*, 117 (1965), 285–306.
- [16] KHINCHIN, A. YA., *Continued Fractions*, 2nd edition. Gosudarstv. Izdat. Techn.-Teor. Lit., Moscow–Leningrad, 1949 (Russian); English translation: The University of Chicago Press, Chicago–London, 1964.
- [17] LANG, S., *Introduction to Diophantine Approximations*, 2nd edition. Springer, New York, 1995.
- [18] MENDÈS FRANCE, M., Principe de la symétrie perturbée, in *Séminaire de Théorie des Nombres* (Paris, 1979–80), pp. 77–98. Progr. Math., 12. Birkhäuser, Boston, MA, 1981.
- [19] MORSE, M. & HEDLUND, G. A., Symbolic dynamics. *Amer. J. Math.*, 60 (1938), 815–866.
- [20] PERRON, O., *Die Lehre von den Kettenbrüchen*. Teubner, Leipzig, 1929.
- [21] QUEFFÉLEC, M., Transcendance des fractions continues de Thue–Morse. *J. Number Theory*, 73 (1998), 201–211.
- [22] — Irrational numbers with automaton-generated continued fraction expansion, in *Dynamical Systems* (Luminy–Marseille, 1998), pp. 190–198. World Sci. Publishing, River Edge, NJ, 2000.
- [23] ROTH, K. F., Rational approximations to algebraic numbers. *Mathematika*, 2 (1955), 1–20; Corrigendum. *Ibid.*, 2 (1955), 168.
- [24] SCHMIDT, W. M., On simultaneous approximations of two algebraic numbers by rationals. *Acta Math.*, 119 (1967), 27–50.
- [25] — Norm form equations. *Ann. of Math.*, 96 (1972), 526–551.
- [26] — *Diophantine Approximation*. Lecture Notes in Math., 785. Springer, Berlin, 1980.
- [27] SHALLIT, J., Real numbers with bounded partial quotients: a survey. *Enseign. Math.*, 38 (1992), 151–187.
- [28] TURING, A. M., On computable numbers, with an application to the Entscheidungsproblem. *Proc. London Math. Soc.*, 42 (1937), 230–265.
- [29] WALDSCHMIDT, M., Un demi-siècle de transcendance, in *Development of Mathematics 1950–2000*, pp. 1121–1186. Birkhäuser, Basel, 2000.

BORIS ADAMCZEWSKI  
 CNRS, Institut Camille Jordan  
 Université Claude Bernard Lyon I  
 Bât. Braconnier  
 21, avenue Claude Bernard  
 FR-69622 Villeurbanne Cedex  
 France  
 boris.adamczewski@math.univ-lyon1.fr

YANN BUGEAUD  
 Université Louis Pasteur  
 U.F.R. de mathématiques  
 7, rue René Descartes  
 FR-67084 Strasbourg Cedex  
 France  
 bugeaud@math.u-strasbg.fr

*Received May 16, 2005*