Palindromic continued fractions

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1. Introduction

An old problem adressed by Khintchin [15] deals with the behaviour of the continued fraction expansion of algebraic real numbers of degree at least three. In particular, it is asked whether such numbers have or not arbitrarily large partial quotients in their continued fraction expansion. Although almost nothing has been proved yet in this direction, some more general speculations are due to Lang [16], including the fact that algebraic numbers of degree at least three should behave like most of the numbers with respect to the Gauss–Khintchin–Kuzmin–Lévy laws. A preliminary step consists in providing explicit examples of transcendental continued fractions. The first result of this type is due to Liouville [17], who constructed real numbers whose sequence of partial quotients grows very fast, too fast for being algebraic. Subsequently, various authors used deeper transcendence criteria from Diophantine approximation to construct other classes of transcendental continued fractions. Of particular interest is the work of Maillet [18] (see also Section 34 of Perron [19]), who was the first to give examples of transcendental continued fractions with bounded partial quotients. Further examples were provided by A. Baker [8, 9], Davison [11], Queffélec [20], Allouche *et al.* [7], Adamczewski and Bugeaud [1, 5], and Adamczewski *et al.* [6], among others. A common feature of all these results is that they apply to real numbers whose continued fraction expansion is 'quasi-periodic' in the sense that it contains arbitrarily long blocks of partial quotients which occur precociously at least twice.

Continued fractions beginning with arbitrarily large palindromes appear in several recent papers [21, 22, 10, 13, 2]. Motivated by this and the general problematic mentioned above, we ask whether precocious occurrences of some symmetric patterns in the continued fraction expansion of an irrational real number do imply that the latter is either quadratic, or transcendental. We obtain three new transendence criteria that apply to a broad class of continued fraction expansions, including expansions with unbounded partial quotients. These results provide the exact counterpart of [1] (see also Theorem 3.1 from [6]), with periodic patterns being replaced by symmetric ones. Like in [1], their proofs heavily depend on the Schmidt Subspace Theorem [24]. As already mentioned, there is a long tradition in using an excess of periodicity to prove the transcendence of some continued fractions. This is indeed very natural: if the continued fraction expansion of the real number ξ begins with, say, the periodic pattern $ABBB$ (here, A, B denote two finite blocks of partial quotients), then ξ is 'very close' to the quadratic irrational real number having the eventually periodic

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continued fraction expansion with preperiod A and period B. The fact that occurrences of symmetric patterns can actually give rise to transcendence statements is more surprising and completely new, though it is already underlying in [22]. It essentially relies on an elementary identity for continued fractions (see Lemma 1 in Section 4).

The present paper is organized as follows. Our transcendence criteria are stated in Section 2 and proved in Sections 5 and 6. In Section 3, we provide an application of one of the transcendence criteria to the explicit construction of transcendental numbers with sharp properties of approximation by rational numbers. All the auxiliary statements are gathered in Section 4.

A previous version of this paper including an application of our results to the transcendence of Maillet–Baker's continued fractions is available on the arXiv at:

`http://arxiv.org/abs/math.NT/0512014`.

This part is removed from the present version since the corresponding results were strongly improved in [5].

2. Main results

Throughout the present work, $\mathcal{A}$ denotes a given set, not necessarily finite. We identify any sequence $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ of elements from $\mathcal{A}$ with the infinite word $a_1 a_2 \dots a_\ell \dots$ Recall that a finite word $a_1 a_2 \dots a_n$ on $\mathcal{A}$ is called a *palindrome* if $a_j = a_{n+1-j}$ for $j = 1, \dots, n$.

Our first transcendence criterion can be stated as follows.

Theorem 1. *Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of positive integers. If the word $\mathbf{a}$ begins in arbitrarily long palindromes, then the real number $\alpha := [0; a_1, a_2, \dots, a_\ell, \dots]$ is either quadratic irrational or transcendental.*

We point out that there is no assumption on the growth of the sequence $(a_\ell)_{\ell \geq 1}$ in Theorem 1, unlike in Theorems 2 and 3 below. This specificity of Theorem 1 is used in Section 3 to construct transcendental continued fractions with a prescribed order of approximation by rational numbers.

Various examples of classical continued fractions turn out to satisfy the assumption of Theorem 1. We display two of them. As shown in [4], if a and b are two distinct positive integers, Theorem 1 provides a short proof of the transcendence of the real number $[a_0; a_1, a_2, \dots\dots]$, whose sequence of partial quotients is the Thue–Morse sequence on the alphabet $\{a, b\}$, that is, with $a_n = a$ (resp. $a_n = b$) if the sum of the binary digits of n is odd (resp. even). This result is originally due to Queffélec [20]. Other interesting examples of continued fractions beginning with arbitrarily large palindromes are the standard Sturmian continued fractions. Given a real number θ with $0 < \theta < 1$ and two distinct positive integers a and b, the standard Sturmian continued fraction of slope θ on the alphabet $\{a, b\}$ is defined by $\xi_\theta := [0; a_1, a_2, \dots]$, where $a_n = a$ if $\lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor = 0$ and $a_n = b$ if $\lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor = 1$. Here, $\lfloor \cdot \rfloor$ denotes the integer part function. The Fibonacci continued fraction $\xi_{(\sqrt{5}-1)/2}$ occurs in the important work of Roy [21, 22]. The reader is directed to [10, 12, 13] for a detailled study of the standard Sturmian continued fractions, which were proved to be transcendental in [7]. Theorem 1 provides an alternative and much shorter proof of the latter result.

Furthermore, it is very easy to construct continued fractions that satisfy the assumption of Theorem 1 and we present now a general and elementary process to do this. Denote the mirror image of a finite word $W := a_1 \dots a_n$ by $\overline{W} := a_n \dots a_1$. In particular, W is a palindrome if and only if $W = \overline{W}$. Given an arbitrary sequence $\mathbf{u} = (U_n)_{n\geq 0}$ of nonempty finite words whose letters are positive integers, we define a sequence of finite words $(A_n)_{n\geq 0}$ by setting $A_0 = U_0$ and $A_{n+1} = A_n U_{n+1} \overline{A_n U_{n+1}}$, for $n \geq 0$. Thus, A_{n+1} begins with A_n and the sequence of finite words $(A_n)_{n\geq 0}$ converges to an infinite word $\mathbf{a} = a_1 a_2 \dots a_\ell \dots$ Actually, every sequence beginning with arbitrarily large palindromes can be constructed in this way. In particular, Theorem 1 can be reformulated as follows.

Corollary 1. *Keep the above notation. If $(a_\ell)_{\ell\geq 1}$ is not eventually periodic, then the real number*

$$\xi_{\mathbf{u}} := [0; a_1, a_2, \dots, a_\ell, \dots]$$

is transcendental.

Before stating our next theorems, we need to introduce some more notation. The length of a finite word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. Recall that a palindrome is a finite word invariant under mirror symmetry (i. e., $W = \overline{W}$). In order to relax this property of symmetry, we introduce the notion of *quasi-palindrome*. For two finite words U and V, the word $UV\overline{U}$ is called a quasi-palindrome of order at most w, where $w = |V|/|U|$. Clearly, the larger w is, the weaker is the property of symmetry. In our next transcendence criterion, we replace the occurrences of aritrarily large palindromes by the ones of arbitrarily large quasi-palindromes of bounded order. However, this weakening of our assumption has a cost, namely, an extra assumption on the growth of the partial quotients is then needed. Fortunately, the latter assumption is not very restrictive. In particular, it is always satisfied by real numbers with bounded partial quotients.

Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(*)$ if $\mathbf{a}$ is not eventually periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded;

(iii) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

We complement Theorem 1 in the following way.

Theorem 2. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded, which is in particular the case when the sequence $\mathbf{a}$ is bounded. If $\mathbf{a}$ satisfies Condition $(*)$, then α is transcendental.

In the statements of Theorems 1 and 2 the palindromes or the quasi-palindromes must appear at the very beginning of the continued fraction under consideration. Fortunately, the ideas used in their proofs allow us to deal also with the more general situation where arbitrarily long quasi-palindromes occur not too far from the beginning.

Let w be a real number. We say that $\mathbf{a}$ satisfies Condition $(*)_w$ if $\mathbf{a}$ is not eventually periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded;
(iii) The sequence $(|U_n|/|W_n|)_{n\geq 1}$ is bounded from below by w;
(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

We are now ready to complement Theorems 1 and 2 as follows.

Theorem 3. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded and set $M = \limsup_{\ell\to+\infty} q_\ell^{1/\ell}$ and $m = \liminf_{\ell\to+\infty} q_\ell^{1/\ell}$. Let w be a real number such that

$$w > 2\frac{\log M}{\log m} - 1. \tag{2.1}$$

If $\mathbf{a}$ satisfies Condition $(*)_w$, then α is transcendental.

We display an immediate consequence of Theorem 3.

Corollary 2. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ converges. If $\mathbf{a}$ satisfies Condition $(*)_w$ for some $w > 1$, then α is transcendental.

Theorems 1 to 3 provide the exact counterparts of Theorems 1 and 2 from [1], with periodic patterns being replaced by symmetric ones. It would be desirable to replace the assumption (2.1) by the weaker one $w > 0$. The statements of Theorems 2 and 3 show that weakening the combinatorial assumption of Theorem 1 needs further assumptions on the size of the partial quotients.

Theorem 2 from [1] was slightly improved in [6], where the following statement was established. Keep the above notation and say that $\mathbf{a}$ satisfies Condition $(**)_w$ if $\mathbf{a}$ is not eventually periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that we have (ii), (iii), (iv) above, together with

(i') For any $n \geq 1$, the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$.

Precisely, Theorem 3.1 from [6] (which slightly improves upon theorem 2 from [1]) asserts that Theorem 3 still holds when **a** satisfies Condition $(**)_w$. This shows that the results obtained when repeated patterns occur are exactly of the same strength as those obtained when symmetric patters occur.

3. Transcendental numbers with prescribed order of approximation

In Satz 6 of [14], Jarník used the continued fraction theory to prove the existence of real numbers with prescribed order of approximation by rational numbers. Let $\varphi : \mathbf{R}_{\geq 1} \to \mathbf{R}_{>0}$ be a positive function. We say that a real number α is 'approximable at order φ' if there exist infinitely many rational numbers p/q with $q > 0$ and $|\alpha - p/q| < \varphi(q)$. Jarník's result can then be stated as follows.

Theorem J. *Let $\varphi : \mathbf{R}_{\geq 1} \to \mathbf{R}_{>0}$ be a non-increasing function such that $\varphi(x) = o(x^{-2})$ as x tends to infinity. Then, there are real numbers α which are approximable at order φ but which are not approximable at any order $c\,\varphi$, with $0 < c < 1$.*

In his proof, Jarník constructed inductively the sequence of partial quotients of α. Actually, he showed that there are uncountably many real numbers α with the required property, thus, in particular, transcendental numbers. However, his construction did not provide any *explicit* example of such a transcendental α.

In the present Section, we apply our Theorem 1 to get, under an extra assumption on the function φ, explicit examples of transcendental numbers satisfying the conclusion of Theorem J.

Theorem 4. *Let $\varphi : \mathbf{R}_{\geq 1} \to \mathbf{R}_{>0}$ be such that $x \mapsto x^2\,\varphi(x)$ is non-increasing and tends to 0 as x tends to infinity. Then, we can construct explicit examples of transcendental numbers α which are approximable at order φ but which are not approximable at any order $c\,\varphi$, with $0 < c < 1$.*

Proof. Throughout the proof, for any real number x, we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x. We will construct inductively the sequence $(b_n)_{n\geq 1}$ of partial quotients of a suitable real number α. Denoting by $(p_n/q_n)_{n\geq 0}$ the sequence of convergents to α, it follows from the continued fraction theory that, for any $n \geq 1$, we have

$$\frac{1}{q_{n-1}^2(b_n+2)} < \left| \alpha - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_{n-1}^2 b_n}. \tag{3.1}$$

Recall that $q_n \geq (3/2)^n$ for any $n \geq 5$. For any $x \geq 1$, set $\Psi(x) = x^2\,\varphi(x)$. Let $n_1 \geq 6$ be such that $\Psi((3/2)^n) \leq 10^{-1}$ for any $n \geq n_1 - 1$. Then, set $b_1 = \ldots = b_{n_1-1} = 1$ and $b_{n_1} = \lceil 1/\Psi(q_{n_1-1}) \rceil$. Observe that $b_{n_1} \geq 10$. Let $n_2 > n_1$ be such that $\Psi((3/2)^n) \leq (10 b_{n_1})^{-1}$ for any $n \geq n_2 - 1$. Then, set $b_{n_1+1} = \ldots = b_{n_2-1} = 1$ and $b_{n_2} = \lceil 1/\Psi(q_{n_2-1}) \rceil$. Observe that $b_{n_2} \geq 10 b_{n_1}$.

At this step, we have

$$\alpha = [0; \overline{1}^{n_1-1}, b_{n_1}, \overline{1}^{n_2-n_1-1}, b_{n_2}, \ldots],$$

where, as in the previous Section, we denote by $\overline{1}^m$ a sequence of m consecutive partial quotients equal to 1. Then, we complete by symmetry, in such a way that the continued fraction expansion of α begins with a palindrome:

$$\alpha = [0; \overline{1}^{n_1-1}, b_{n_1}, \overline{1}^{n_2-n_1-1}, b_{n_2}, \overline{1}^{n_2-n_1-1}, b_{n_1}, \overline{1}^{n_1-1}, \ldots].$$

At this stage, we have constructed the first $2n_2-1$ partial quotients of α. Let $n_3 > 2n_2$ be such that $\Psi((3/2)^n) \le (10 b_{n_2})^{-1}$ for any $n \ge n_3 - 1$. Then, set $b_{2n_2} = \ldots = b_{n_3-1} = 1$ and $b_{n_3} = \lceil 1/\Psi(q_{n_3-1}) \rceil$. Observe that $b_{n_3} \ge 10 b_{n_2}$. Then, we again complete by symmetry, and we repeat our process in order to define n_4, b_{n_4}, and so on.

Clearly, the real number constructed in this way begins with infinitely many palindromes, thus it is either quadratic or transcendental by Theorem 1. Moreover, the assumption on the function φ implies that α has unbounded partial quotients. It thus follows that it is transcendental. It remains for us to prove that it has the required property of approximation.

By (3.1), for any $j \ge 1$, we have

$$\frac{\varphi(q_{n_j-1})}{1 + 3\, q_{n_j-1}^2\, \varphi(q_{n_j-1})} < \left| \alpha - \frac{p_{n_j-1}}{q_{n_j-1}} \right| < \varphi(q_{n_j-1}). \tag{3.2}$$

Let p_n/q_n with $n \ge n_2$ be a convergent to α not in the subsequence $(p_{n_j-1}/q_{n_j-1})_{j \ge 1}$, and let k be the integer defined by $n_k - 1 < n < n_{k+1} - 1$. Then, by combining (3.1) with $b_{n+1} \le b_{n_{k-1}}$, we have

$$\begin{aligned} \left| \alpha - \frac{p_n}{q_n} \right| &> \frac{1}{q_n^2 (b_{n+1} + 2)} \ge \frac{1}{q_n^2 (b_{n_{k-1}} + 2)} \\ &\ge \frac{1}{3 q_n^2\, b_{n_{k-1}}} \ge \frac{\varphi(q_n)}{3 q_{n_k-1}^2\, \varphi(q_{n_k-1})\, b_{n_{k-1}}}, \end{aligned} \tag{3.3}$$

since $x \mapsto x^2 \varphi(x)$ is non-increasing. We then infer from (3.3) and

$$b_{n_{k-1}} \le \frac{b_{n_k}}{10} \le \frac{11}{100} \cdot \frac{1}{q_{n_k-1}^2\, \varphi(q_{n_k-1})}$$

that

$$\left| \alpha - \frac{p_n}{q_n} \right| \ge 3\, \varphi(q_n). \tag{3.4}$$

To conclude, we observe that it follows from (3.2) that α is approximable at order φ, and from (3.2), (3.4) and the fact that φ is non-increasing that α is not approximable at any order $c\varphi$ with $0 < c < 1$. The proof of Theorem 4 is complete. □

4. Auxiliary results

The proofs of Theorems 2 and 3 depend on a deep result from Diophantine approximation, namely the powerful Schmidt Subspace Theorem, stated as Theorem B below. However, we do not need the full force of this theorem to prove our Theorem 1: the transcendence criterion given by Theorem A is sufficient for our purpose.

Theorem A. (W. M. Schmidt). *Let α be a real number, which is neither rational, nor quadratic. If there exist a real number $w > 3/2$ and infinitely many triples of integers (p, q, r) with $q > 0$ such that*

$$\max\left\{\left|\alpha - \frac{p}{q}\right|, \left|\alpha^2 - \frac{r}{q}\right|\right\} < \frac{1}{q^w},$$

then α is transcendental.

Proof : See [23]. □

Theorem B. (W. M. Schmidt). *Let $m \geq 2$ be an integer. Let $L_1, \ldots, L_m$ be linearly independent linear forms in $\mathbf{x} = (x_1, \ldots, x_m)$ with algebraic coefficients. Let ε be a positive real number. Then, the set of solutions $\mathbf{x} = (x_1, \ldots, x_m)$ in $\mathbf{Z}^m$ to the inequality*

$$|L_1(\mathbf{x}) \ldots L_m(\mathbf{x})| \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbf{Q}^m$.

Proof : See e.g. [24] or [25]. □

For the reader convenience, we further recall some well-known results from the theory of continued fractions, whose proofs can be found e.g. in the book of Perron [19]. The seemingly innocent Lemma 1 appears to be crucial in the proofs of Theorems 2 to 4.

Lemma 1. *Let $\alpha = [0; a_1, a_2, \ldots]$ be a real number with convergents $(p_\ell/q_\ell)_{\ell \geq 1}$. Then, for any $\ell \geq 2$, we have*

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1].$$

Lemma 2. *Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers. Let $n \geq 1$ such that $a_i = b_i$ for any $i = 1, \ldots, n$. We then have $|\alpha - \beta| \leq q_n^{-2}$, where q_n denotes the denominator of the n-th convergent to α.*

Lemma 3. *Let $n \geq 2$ be an integer. For any positive integers $a_1, \ldots, a_n$, the denominator of the rational number $[0; a_1, \ldots, a_n]$ is at least equal to $(\sqrt{2})^n$.*

For positive integers $a_1, \ldots, a_m$, we denote by $K_m(a_1, \ldots, a_m)$ the denominator of the rational number $[0; a_1, \ldots, a_m]$. It is commonly called a *continuant.*

Lemma 4. *For any positive integers $a_1, \ldots, a_m$ and any integer k with $1 \leq k \leq m - 1$, we have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) &\leq K_m(a_1, \ldots, a_m) \\ &\leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m). \end{aligned}$$

5. Proof of Theorem 1

Let n be a positive integer. Denote by p_n/q_n the n-th convergent to α, that is, $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$. By the theory of continued fraction, we have

$$M_n := \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Since such a decomposition is unique, the matrix M_n is symmetrical if and only if the word $a_1 a_2 \ldots a_n$ is a palindrome. Assume that this is case. Then, we have $p_n = q_{n-1}$. Recalling that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2} \quad \text{and} \quad \left| \alpha - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_{n-1}^2},$$

we infer from $0 < \alpha < 1$, $a_1 = a_n$, $|p_n q_{n-1} - p_{n-1} q_n| = 1$ and $q_n \le (a_n + 1) q_{n-1}$ that

$$\begin{aligned} \left| \alpha^2 - \frac{p_{n-1}}{q_n} \right| &\le \left| \alpha^2 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_n}{q_n} \right| \le \left| \alpha + \frac{p_{n-1}}{q_{n-1}} \right| \cdot \left| \alpha - \frac{p_n}{q_n} \right| + \frac{1}{q_n q_{n-1}} \\ &\le 2 \left| \alpha - \frac{p_n}{q_n} \right| + \frac{1}{q_n q_{n-1}} < \frac{a_1 + 3}{q_n^2}, \end{aligned}$$

whence

$$\max \left\{ \left| \alpha - \frac{p_n}{q_n} \right|, \left| \alpha^2 - \frac{p_{n-1}}{q_n} \right| \right\} < \frac{a_1 + 3}{q_n^2}. \tag{5.1}$$

Consequently, if the sequence of the partial quotients of α begins in arbitrarily long palindromes, then (5.1) is satisfied for infinitely many integer triples (p_n, q_n, p_{n-1}). By Theorem A, this shows that α is either quadratic or transcendental. $\square$

6. Proofs of Theorems 2 and 3

Throughout the proofs of Theorems 2 and 3, for any finite word $U = u_1 \ldots u_n$ on $\mathbf{Z}_{\ge 1}$, we denote by $[0; U]$ the rational number $[0; u_1, \ldots, u_n]$.

Proof of Theorem 2. Keep the notation and the hypothesis of this theorem. Let $(U_n)_{n \ge 1}$ and $(V_n)_{n \ge 1}$ be the sequences occurring in the definition of Condition $(*)$. Set $r_n = |U_n|$ and $s_n = |U_n V_n \overline{U}_n|$, for any $n \ge 1$. We want to prove that the real number

$$\alpha := [0; a_1, a_2, \ldots]$$

is transcendental. By assumption, we already know that α is irrational and not quadratic. Therefore, we assume that α is algebraic of degree at least three and we aim at deriving a contradiction.

Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to α. The key fact for the proof of Theorem 2 is the equality

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1],$$

given by Lemma 1. In other words, if W_ℓ denotes the prefix of length ℓ of the sequence $\mathbf{a}$, then $q_{\ell-1}/q_\ell = [0; \overline{W_\ell}]$. Since, by assumption, we have

$$\frac{p_{s_n}}{q_{s_n}} = [0; U_n V_n \overline{U_n}],$$

we get that

$$\frac{q_{s_n-1}}{q_{s_n}} = [0; U_n \overline{V_n}\ \overline{U_n}],$$

and it follows from Lemma 2 that

$$|q_{s_n}\alpha - q_{s_n-1}| < q_{s_n} q_{r_n}^{-2}. \tag{6.1}$$

This shows in particular that

$$\lim_{n\to+\infty} \frac{q_{s_n-1}}{q_{s_n}} = \alpha. \tag{6.2}$$

Furthermore, we clearly have

$$|q_{s_n}\alpha - p_{s_n}| < q_{s_n}^{-1} \quad \text{and} \quad |q_{s_n-1}\alpha - p_{s_n-1}| < q_{s_n}^{-1}. \tag{6.3}$$

Consider now the four linearly independent linear forms with algebraic coefficients:

$$\begin{aligned}
L_1(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_3,\\
L_2(X_1, X_2, X_3, X_4) =& \alpha X_2 - X_4,\\
L_3(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_2,\\
L_4(X_1, X_2, X_3, X_4) =& X_2.
\end{aligned}$$

Evaluating them on the quadruple $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$, it follows from (6.1) and (6.3) that

$$\prod_{1\leq j\leq 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| < q_{r_n}^{-2}. \tag{6.4}$$

Our assumption and Lemma 3 imply that there exists a real number M such that

$$\sqrt{2} \leq q_\ell^{1/\ell} \leq M$$

for any integer $\ell \geq 3$. Thus, for any integer $n \geq 3$, we have

$$q_{r_n} \geq \sqrt{2}^{r_n} = (M^{s_n})^{(r_n \log\sqrt{2})/(s_n \log M)} \geq q_{s_n}^{(r_n \log\sqrt{2})/(s_n \log M)}$$

and we infer from (6.4) and from (ii) of Condition (∗) that

$$\prod_{1\le j\le 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| < q_{s_n}^{-\varepsilon}$$

holds with

$$\varepsilon = (2\log M)^{-1} \cdot (2 + \limsup_{n\to+\infty} |V_n|/|U_n|)^{-1}$$

for every sufficiently large n.

It then follows from Theorem B that the points $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$ lie in a finite number of proper subspaces of $\mathbf{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1 q_{s_n} + x_2 q_{s_n-1} + x_3 p_{s_n} + x_4 p_{s_n-1} = 0, \tag{6.5}$$

for any n in $\mathcal{N}_1$. Dividing (6.5) by q_{s_n}, we obtain

$$x_1 + x_2 \frac{q_{s_n-1}}{q_{s_n}} + x_3 \frac{p_{s_n}}{q_{s_n}} + x_4 \frac{p_{s_n-1}}{q_{s_n-1}} \cdot \frac{q_{s_n-1}}{q_{s_n}} = 0. \tag{6.6}$$

By letting n tend to infinity along $\mathcal{N}_1$ in (6.6), it follows from (6.2) that

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0.$$

Since, by assumption, α is not a quadratic number, we have $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, (6.5) implies that

$$q_{s_n-1} = p_{s_n}. \tag{6.7}$$

Consider now the three linearly independent linear forms with algebraic coefficients:

$$L'_1(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_2, \quad L'_2(Y_1, Y_2, Y_3) = \alpha Y_2 - Y_3, \quad L'_3(Y_1, Y_2, Y_3) = Y_1.$$

Evaluating them on the triple $(q_{s_n}, p_{s_n}, p_{s_n-1})$, we infer from (6.3) and (6.7) that

$$\prod_{1\le j\le 3} |L'_j(q_{s_n}, p_{s_n}, p_{s_n-1})| < q_{s_n}^{-1}.$$

It then follows from Theorem B that the points $(q_{s_n}, p_{s_n}, p_{s_n-1})$ with n in $\mathcal{N}_1$ lie in a finite number of proper subspaces of $\mathbf{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$y_1 q_{s_n} + y_2 p_{s_n} + y_3 p_{s_n-1} = 0, \tag{6.8}$$

for any n in $\mathcal{N}_2$. Dividing (6.8) by q_{s_n}, we get

$$y_1 + y_2 \frac{p_{s_n}}{q_{s_n}} + y_3 \frac{p_{s_n-1}}{q_{s_n-1}} \cdot \frac{q_{s_n-1}}{q_{s_n}} = 0. \tag{6.9}$$

By letting n tend to infinity along $\mathcal{N}_2$, it thus follows from (6.7) and (6.9) that

$$y_1 + y_2\alpha + y_3\alpha^2 = 0.$$

Since (y_1, y_2, y_3) is a non-zero triple of integers, we have reached a contradiction. Consequently, the real number α is transcendental. This completes the proof of the theorem. $\square$

Proof of Theorem 3. Keep the notation and the hypothesis of this theorem. Assume that the parameter w is fixed, as well as the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$. Set also $r_n = |W_n|$, $s_n = |W_n U_n|$ and $t_n = |W_n U_n V_n \overline{U_n}|$, for any $n \geq 1$. We want to prove that the real number

$$\alpha := [0; a_1, a_2, \ldots]$$

is transcendental. By assumption, we already know that α is irrational and not quadratic. Therefore, we assume that α is algebraic of degree at least three and we aim at deriving a contradiction. Throughout this Section, the constants implied by $\ll$ depend only on α. In view of Theorem 2, we may assume that $r_n \geq 1$ for any n.

The key idea for our proof is to consider, for any positive integer n, the rational P_n/Q_n defined by

$$\frac{P_n}{Q_n} := [0; W_n U_n V_n \overline{U_n}\, \overline{W_n}]$$

and to use the fact that the word $W_n U_n V_n \overline{U_n}\, \overline{W_n}$ is a quasi-palindrome. Let P'_n/Q'_n denote the last convergent to P_n/Q_n and different from P_n/Q_n. By assumption we have

$$\frac{p_{t_n}}{q_{t_n}} = [0; W_n U_n V_n \overline{U_n}]$$

and it thus follows from Lemma 2 that

$$|Q_n\alpha - P_n| < Q_n q_{t_n}^{-2} \tag{6.10}$$

and

$$|Q'_n\alpha - P'_n| < Q'_n q_{t_n}^{-2}, \tag{6.11}$$

since $\overline{W}_n$ has at least one letter. Furthermore, Lemma 1 implies that

$$\frac{Q'_n}{Q_n} = [0; W_n U_n \overline{V_n}\, \overline{U_n}\, \overline{W_n}],$$

and we get from Lemma 2 that

$$|Q_n\alpha - Q'_n| < Q_n q_{s_n}^{-2}. \tag{6.12}$$

This shows in particular that

$$\lim_{n\to+\infty} \frac{Q'_n}{Q_n} = \alpha. \tag{6.13}$$

Consider now the following four linearly independent linear forms with algebraic coefficients:

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_3, \\ L_2(X_1, X_2, X_3, X_4) =& \alpha X_2 - X_4, \\ L_3(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_2, \\ L_4(X_1, X_2, X_3, X_4) =& X_2. \end{aligned}$$

Evaluating them on the quadruple (Q_n, Q'_n, P_n, P'_n), it follows from (6.10), (6.11) and (6.12) that

$$\prod_{1 \le j \le 4} |L_j(Q_n, Q'_n, P_n, P'_n)| < Q_n^4 q_{t_n}^{-4} q_{s_n}^{-2}. \tag{6.14}$$

We infer from Lemma 4 that

$$q_{t_n} q_{r_n} \le Q_n \le 2 q_{t_n} q_{r_n} \quad \text{and} \quad q_{s_n}^2 \le Q_n \le q_{t_n}^2, \tag{6.15}$$

and thus (6.14) gives

$$\prod_{1 \le j \le 4} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll q_{r_n}^4 \, q_{s_n}^{-2}. \tag{6.16}$$

Moreover, by our assumption (2.1), there exists $\eta > 0$ such that, for any n large enough, we have

$$|U_n| \ge \left(2 \, \frac{\log M}{\log m} \cdot \frac{1+\eta}{1-\eta} - 1 \right) |W_n|,$$

thus

$$s_n \ge \frac{2(1+\eta)(\log M)}{(1-\eta)(\log m)} \, r_n.$$

Consequently, assuming that n is sufficiently large, we get

$$m^{(1-\eta) s_n} \ge M^{2(1+\eta) r_n}$$

and

$$q_{s_n} \ge q_{r_n}^{2+\eta'},$$

for some positive real number η'. It then follows from (6.16) that

$$\prod_{1 \le j \le 4} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll q_{s_n}^{-2\eta'/(2+\eta')}. \tag{6.17}$$

Our assumption and Lemma 3 imply that we have

$$\sqrt{2} \le q_\ell^{1/\ell} \le 2M,$$

for any ℓ large enough. Thus, for any integer n large enough, we have

$$\begin{aligned} q_{s_n} \ge \sqrt{2}^{s_n} = ((2M)^{t_n})^{(s_n \log \sqrt{2})/(t_n \log 2M)} &\ge q_{t_n}^{(s_n \log \sqrt{2})/(t_n \log 2M)} \\ &\ge Q_n^{(s_n \log \sqrt{2})/(2 t_n \log 2M)}, \end{aligned}$$

by (6.15). We then infer from (6.17) and from (ii) of Condition $(*)_w$ that

$$\prod_{1\le j\le 4} |L_j(Q_n, Q_n', P_n, P_n')| \ll Q_n^{-\varepsilon} \tag{6.18}$$

holds for some positive ε.

It then follows from Theorem B that the points (Q_n, Q_n', P_n, P_n') lie in a finite number of proper subspaces of $\mathbf{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_3$ such that

$$x_1 Q_n + x_2 Q_n' + x_3 P_n + x_4 P_n' = 0, \tag{6.19}$$

for any n in $\mathcal{N}_3$. Dividing by Q_n, we obtain

$$x_1 + x_2 \frac{Q_n'}{Q_n} + x_3 \frac{P_n}{Q_n} + x_4 \frac{P_n'}{Q_n'} \cdot \frac{Q_n'}{Q_n} = 0.$$

By letting n tend to infinity along $\mathcal{N}_3$, we infer from (6.13) that

$$x_1 + (x_2 + x_3)\alpha + x_4 \alpha^2 = 0.$$

Since $(x_1, x_2, x_3, x_4) \neq (0,0,0,0)$ and since α is irrational and not quadratic, we have $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, (6.19) implies that

$$Q_n' = P_n. \tag{6.20}$$

Consider now the following three linearly independent linear forms with algebraic coefficients:

$$L_1'(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_2,\ L_2'(Y_1, Y_2, Y_3) = \alpha Y_2 - Y_3,\ L_3'(Y_1, Y_2, Y_3) = Y_1.$$

Evaluating them on the quadruple (Q_n, P_n, P_n'), it follows from (6.10), (6.11), (6.15) and (6.20) that

$$\prod_{1\le j\le 3} |L_j(Q_n, P_n, P_n')| \ll Q_n^3 q_{t_n}^{-4} \ll q_{r_n}^4 Q_n^{-1} \ll q_{r_n}^4 q_{s_n}^{-2} \ll Q_n^{-\varepsilon},$$

with the same ε as in (6.18). It then follows from Theorem B that the points (Q_n, P_n, P_n') lie in a finite number of proper subspaces of $\mathbf{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_4$ such that

$$y_1 Q_n + y_2 P_n + y_3 P_n' = 0,$$

for any n in $\mathcal{N}_4$. We then proceed exactly as at the end of the proof of Theorem 2 to reach a contradiction. This finishes the proof of our theorem. □

7. Conclusion

As already mentioned in Section 2, the transcendence criteria obtained in the present paper by making use of symmetric patterns in continued fractions are analogous to those derived thanks to repetitive patterns in [1, 6]. Moreover, all these results rely on a common tool: the Subspace Theorem. This naturally leads us to ask whether it would be possible to derive some results of the present paper from [1, 6] or vice-versa; or more generally whether there is a strong link between occurrences of repetitive patterns and those of symmetric ones in continued fractions. We shall now show that, in general, such a link fails.

Let us consider the free monoid generated by the alphabet $\{1,2,3\}$ and the morhism σ mapping 1 on 123, 2 on 123 and 3 on 1. Iterating this morphism from the letter 1 gives rise to an infinite sequence and then to the following continued fraction

$$[0;1,2,3,1,2,3,1,1,2,3,1,2,3,1,1,2,3,1,2,3,1,2,3,1,...].$$

This continued fraction contains a lot of large repetitive patterns. In particular, it begins with arbitrarily long squares (blocks of the form AA), so that it is transcendental by virtue of Theorem 1 of [1]. On the other hand, it contains no large symmetric pattern since the pattern $3,2$ does not occur at all. Hence, there is no reason that occurrences of repetitive patterns do imply those of symmetric ones.

Now, let us have a look on the converse situation which is more delicate. For every positive integers $n<m$, let us denote by $X_{[n,m]}$ the pattern $n,n+1,n+2,\dots,m$. We then define the continued fraction

$$\xi := [0, X_{[1,2]}, \overline{X_{[1,2]}}, X_{[3,8]}, \overline{X_{[3,8]}}, X_{[9,64]}, \overline{X_{[9,64]}}, \dots, X_{[8^n+1,8^{n+1}]}, \overline{X_{[8^n+1,8^{n+1}]}}, \dots].$$

The transcendence of ξ follows from the proof of our Theorem 3 thanks to the precocious occurrences of large palindromes, namely $X_{[8^n+1,8^{n+1}]}, \overline{X_{[8^n+1,8^{n+1}]}}$. To see this, for any integer $n\geq 2$, set

$$W_n = X_{[1,2]}\overline{X_{[1,2]}}, X_{[3,8]}\overline{X_{[3,8]}}X_{[9,64]}\overline{X_{[9,64]}}, \dots, X_{[8^{n-1}+1,8^n]}, \overline{X_{[8^{n-1}+1,8^n]}},$$

$$U_n = X_{[8^n+1,8^{n+1}]}, \quad V_n = 0,$$

and define s_n, Q_n as in the proof of Theorem 3. We infer from Lemma 4 that $q_{s_n} \gg Q_n^{1/2}$, thus (6.18) holds for some positive ε. Continuing exactly as in the proof of Theorem 3, we get that ξ is transcendental. We point out that the continued fraction expansion of ξ has no large repetitive pattern, so that occurrences of symmetric patterns do not imply in general those of repetitive ones.

Despite of the previous example, there is an important case where "symmetry implies periodicity". In more concrete terms, the following statement holds. Let us assume that $\mathbf{a}$ is a bounded sequence of positive integers beginning with arbitrarily large palindromes W_n. If we add the condition that $\mathbf{a}$ has a positive palindrome density, that is, if

$$\limsup_{n\to+\infty} \frac{|W_{n+1}|}{|W_n|} < +\infty,$$

then the sequence **a** contains arbitrarily large initial repetitions, in the sense that it satisfies the condition $(*)_w$ of Theorem 1 of [1] for some real number $w > 1$. Such relation is for instance used in the proof of Theorem 4 of [3].

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References

[1] B. Adamczewski & Y. Bugeaud, On the complexity of algebraic numbers, II. Continued fractions, *Acta Math.* **195** (2005), 1–20.

[2] B. Adamczewski & Y. Bugeaud, On the Littlewood conjecture in simultaneous Diophantine approximation, *J. London Math. Soc.* **73** (2006), 355–366.

[3] B. Adamczewski & Y. Bugeaud, Real and p-adic expansions involving symmetric patterns, Int. Math. Res. Not. (2006), Art. ID 75968, 17 pp.

[4] B. Adamczewski & Y. Bugeaud, A Short Proof of the Transcendence of Thue-Morse Continued Fractions, *Amer. Math. Monthly*, to appear.

[5] B. Adamczewski & Y. Bugeaud, On the Maillet-Baker continued fractions, *J. Reine Angew. Math.*, to appear.

[6] B. Adamczewski, Y. Bugeaud & L. Davison, Continued fractions and transcendental numbers, *Ann. Inst. Fourier*, to appear.

[7] J.-P. Allouche, J. L. Davison, M. Queffélec & L. Q. Zamboni, Transcendence of Sturmian or morphic continued fractions, *J. Number Theory* **91** (2001), 39–66.

[8] A. Baker, Continued fractions of transcendental numbers, *Mathematika* **9** (1962), 1–8.

[9] A. Baker, On Mahler's classification of transcendental numbers, *Acta Math.* **111** (1964), 97–120.

[10] Y. Bugeaud & M. Laurent, Exponents of Diophantine and Sturmian continued fractions, *Ann. Inst. Fourier* **55** (2005) 773–804.

[11] J. L. Davison, A class of transcendental numbers with bounded partial quotients, in *Number Theory and Applications*, R. A. Mollin, ed., Kluwer Academic Publishers, 1989, pp. 365–371.

[12] S. Fischler, Palindromic Prefixes and Episturmian Words, *J. Combin. Theory, Ser. A* **113** (2006), 1281–1304

[13] S. Fischler, Palindromic Prefixes and Diophantine Approximation, *Monatsh. Math.*, to appear.

[14] V. Jarník, Über die simultanen Diophantische Approximationen, *Math. Z.* **33** (1931), 505–543.

[15] A. Ya. Khintchine, *Continued Fractions*, 2nd ed., Gosudarstv. Izdat. Tehn.-Theor. Lit. Moscow-Leningrad, 2nd edition, 1949 (Russian); English translation: The University of Chicago Press, Chicago-London, 1964.

[16] S. Lang, *Introduction to Diophantine Approximations*, 2nd ed., Springer-Verlag, New-York, 1995.

[17] J. Liouville, Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationelles algébriques, *C. R. Acad. Sci. Paris* **18** (1844), 883–885 and 993–995.

[18] E. Maillet, *Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions*, Gauthier-Villars, Paris, 1906.

[19] O. Perron, *Die Lehre von den Ketterbrüchen*, Teubner, Leipzig, 1929.

[20] M. Queffélec, Transcendance des fractions continues de Thue–Morse, *J. Number Theory* **73** (1998), 201–211.

[21] D. Roy, Approximation to real numbers by cubic algebraic integers, II, *Ann. of Math. (2)* **158** (2003) 1081–1087.

[22] D. Roy, Approximation to real numbers by cubic algebraic integers, I, *Proc. London Math. Soc. (3)* **88** (2004) 42–62.

[23] W. M. Schmidt, On simultaneous approximations of two algebraic numbers by rationals, *Acta Math.* **119** (1967), 27–50.

[24] W. M. Schmidt, Norm form equations, *Ann. of Math.* **96** (1972), 526–551.

[25] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics **785**, Springer-Verlag, Berlin, 1980.

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