MAHLER'S METHOD IN SEVERAL VARIABLES AND FINITE AUTOMATA

BORIS ADAMCZEWSKI AND COLIN FAVERJON

Abstract. We develop a theory of linear Mahler systems in several variables from the perspective of transcendence and algebraic independence, which also includes the possibility of dealing with several systems associated with sufficiently independent matrix transformations. Our main results go far beyond the existing literature, also surpassing those of two unpublished preprints the authors made available on the arXiv in 2018. The main new feature is that they apply now without any restriction on the matrices defining the corresponding Mahler systems. As a consequence, we settle several problems concerning expansions of numbers in multiplicatively independent bases. For instance, we prove that no irrational real number can be automatic in two multiplicatively independent integer bases, and we give a new proof and a broad algebraic generalization of Cobham's theorem in automata theory. We also provide a new proof and a multivariate generalization of Nishioka's theorem, a landmark result in Mahler's method.

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1. Introduction

It is commonly expected that expansions of numbers in multiplicatively independent bases, such as 2 and 10, should have no common structure. However, it seems extraordinarily difficult to confirm this naive heuristic principle in some way or another. In the late 1960s, Furstenberg [26, 27]

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suggested a series of conjectures, which became famous, and aim to capture this heuristic (see Conjecture A.1 in Appendix A). Despite recent remarkable progress by Shmerkin [58] and Wu [61], Conjecture A.1 remains totally out of reach of the current methods. As always when mathematicians have to face such an enormous gap between heuristic and knowledge, it becomes essential to find out *good problems*. By that, we mean problems which, on the one hand, formalize and express the general heuristic, and, on the other hand, whose solution does not seem desperately out of reach. While Furstenberg's conjectures take place in a dynamical setting, we use instead the language of automata theory to formulate some related conjectures that, hopefully, belong to the above category. These conjectures are introduced and discussed in Appendix A. Thanks to the work of Cobham [20], various problems involving numbers generated by finite automata can be translated and extended to problems concerning transcendence and algebraic independence of values of M-functions. Furthermore, such problems fall into the scope of *Mahler's method*. In the end, we are able to settle our conjectures after proving the apparently unrelated Theorem 1.1.

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ denote the field of algebraic numbers and, given a field $\mathbb{K} \subset \mathbb{C}$, let $\mathbb{K}\{z\}$ denote the ring of convergent power series with coefficients in $\mathbb{K}$. Given an integer $q \geq 2$, $f(z) \in \overline{\mathbb{Q}}\{z\}$ is said to be a *q-Mahler function* if there exist polynomials $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_m(z)f(z^{q^m}) \;=\; 0. \tag{1.1}$$

If $f(z)$ is q-Mahler for some q, we simply say that $f(z)$ is a Mahler function, or an M-function. The coefficients of an M-function generate only a finite field extension of $\mathbb{Q}$. Let us also recall that nonzero complex numbers $x_1, \ldots, x_r$ are multiplicatively independent if there is no nonzero tuple of integers $(n_1, \ldots, n_r)$ such that $x_1^{n_1} \cdots x_r^{n_r} = 1$.

Theorem 1.1. *Let $r \geq 1$ be an integer and $\mathbb{K} \subseteq \overline{\mathbb{Q}}$ be a field. For every integer i, $1 \leq i \leq r$, we let $q_i \geq 2$ be an integer, $f_i(z) \in \mathbb{K}\{z\}$ be a q_i-Mahler function, and $\alpha_i \in \mathbb{K}$, $0 < |\alpha_i| < 1$, be such that $f_i(z)$ is well-defined at α_i. Let us assume that one of the two following properties holds.*

(i) *The numbers $\alpha_1, \ldots, \alpha_r$ are multiplicatively independent.*

(ii) *The numbers $q_1, \ldots, q_r$ are pairwise multiplicatively independent.*

Then $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them belongs to $\mathbb{K}$.

Until now, Theorem 1.1 was only proved when $r = 1$. This special case, conjectured by Cobham [20] in 1968 and settled by the authors [8] in 2017, implies that the decimal expansion of algebraic irrational numbers cannot be generated by finite automata[1]. Also, an algorithm to determine whether the numbers $f_i(\alpha_i)$ belong to $\mathbb{K}$ or not is described in [9].

Let us point out the four main difficulties we have to face when trying to prove Theorem 1.1.

(i) We have to consider a bunch of *arbitrary M-functions*. In contrast, many results in the past where restricted to inhomogeneous

[1]This result was first proved by Bugeaud and the first author [4] by means of the subspace theorem.

order one equations (see Section 2). That is, equations of the form $p_{-1}(z)+p_0(z)f(z)+p_1(z)f(z^q) = 0$. Being able to deal with arbitrary equations becomes essential for applications involving automata.

(ii) Given an M-function, we have to consider its values at *arbitrary algebraic points* where it is well-defined, while a classical feature of results in this framework is that they are only available for points which are *regular*[2] with respect to the underlying Mahler system.

(iii) We have to consider simultaneously values of M-functions at *different algebraic points*. In the setting of Siegel E-functions, the study of algebraic relations between values of E-functions at different algebraic points can be achieved by considering different E-functions at the same point. Indeed, if $f(z)$ is an E-function and α is an algebraic number, then the function $f(\alpha z)$ is still an E-function. However, this trick no longer works for M-functions.

(iv) We have to consider M-functions associated with *different transformations* (*i.e.*, $z \mapsto z^q$ with different q).

Thanks to the work of Ku. Nishioka [49], the transcendence theory of linear Mahler systems in one variable is well-developed. It has even reached a rather definitive stage after the recent works of Philippon [53] and the authors [8]. These new results provide tools to overcome (ii), and also (i) in some situations. However, Theorem 1.1 does not fall into the scope of Mahler's method in one variable. In particular, the problem raised by (iii) requires a major development of Mahler's method in several variables. Partial results in this direction are due to Mahler [42], Kubota [31], Loxton and van der Poorten [37, 39], and Nishioka [51]. Last but not least, (iv) is a source of well-known difficulties and only limited results, though of great interest, have been obtained by Nishioka [50] and Masser [45].

In what follows, we develop a theory of linear Mahler systems in several variables from the perspective of transcendence and algebraic independence, which also includes the possibility of dealing with several systems associated with sufficiently independent matrix transformations. It is condensed in three main general results, Theorems 2.3, 2.6, and 2.8, which go far beyond the existing literature. These results also surpass those of two unpublished preprints [10, 11] the authors made available on the arXiv in 2018. The main new feature with respect to these two preprints is that our results applies now without any restriction on the matrices defining the systems under consideration. *In fine*, the new approach we follow allows us to overcome all the aforementioned difficulties. To tell the truth, proving Conjectures A.2 and A.3, and Corollary A.4 stated in Appendix A, was our initial goal. In order to measure the relevance of the theory eventually developed in Section 2 to reach this goal, the reader is thus encouraged to look at Appendix A. However, in our opinion, this theory is equally valuable in its own right.

Organization of the paper. In Section 2, we state our main results concerning the study of linear Mahler systems in several variables, namely Theorems 2.3, 2.6, and 2.8. We also discuss the three main new ingredients of our approach in Section 2.4. Some notation are introduced in Section 3. As

[2]See Definition 2.2.

made clear in Section 2, the strength of our results strongly depends on our ability to provide simple and natural conditions that ensure certain admissibility conditions. This problem is addressed in Section 4 where concrete and optimal conditions are given. In Section 5 we prove a new vanishing theorem that is a key ingredient for proving Theorem 2.8. In Section 6, we state Theorem 6.2, a general result dealing with families of linear Mahler systems associated with sufficiently independent transformations. Some preliminary results for proving Theorem 6.2 are gathered in Section 7. Then Theorem 6.2 is proved in Section 8, while Theorems 2.3, 2.6, 2.8, and Corollaries 2.5 and 2.9 are derived from Theorem 6.2 in Section 9. Finally, we deduce Theorem 1.1 from Theorems 2.6 and 2.8 in Section 10, and Conjectures A.2, A.3, and Corollary A.4 from Theorem 1.1 in Appendix A.

2. Mahler's method in several variables

Let $n \geq 1$ be an integer and $\boldsymbol{z} = (z_1, \ldots, z_n)$ be a n-tuple of indeterminates. We let $\overline{\mathbb{Q}}\{\boldsymbol{z}\}$ denote the ring of n variables convergent power series with algebraic coefficients, and we set $\overline{\mathbb{Q}}^\star := \overline{\mathbb{Q}} \setminus \{0\}$. Given a field extension $\mathbb{L}$ of a field $\mathbb{K}$, and $a_1, \ldots, a_m$ in $\mathbb{L}$, we let $\mathrm{tr.deg}_{\mathbb{K}}(a_1, \ldots, a_m)$ denote the transcendence degree of $\mathbb{K}(a_1, \ldots, a_m)$ over $\mathbb{K}$.

Given $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ related by a system of the form (2.2) (see below) and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ a point at which these functions are well-defined, Mahler's method aims at transferring results about the absence of algebraic (resp., linear) relations between $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ over $\overline{\mathbb{Q}}(\boldsymbol{z})$ to the absence of algebraic (resp., linear) relations over $\overline{\mathbb{Q}}$ between the complex numbers $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$. In particular, a reoccurring theme consists in establishing the equality

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) , \tag{2.1}$$

under some reasonable assumptions on $A(\boldsymbol{z})$, T, and $\boldsymbol{\alpha}$. This problem goes back to the pioneering work of Mahler [40, 41, 42] at the end of the 1920s.

2.1. Mahler's transformations and linear Mahler systems.

Let $T = (t_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with nonnegative integer coefficients. We set

$$T\boldsymbol{z} = (z_1^{t_{1,1}} z_2^{t_{1,2}} \cdots z_n^{t_{1,n}}, \ldots, z_1^{t_{n,1}} z_2^{t_{n,2}} \cdots z_n^{t_{n,n}}) ,$$

and we let also T act on $\mathbb{C}^n$ in a similar way.

Definition 2.1. A *linear T-Mahler system*, or simply a *Mahler system*, is a system of functional equations of the form

$$\begin{pmatrix} f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = A(\boldsymbol{z}) \begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix} , \tag{2.2}$$

where $A(\boldsymbol{z}) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$. A *Mahler function* $f(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ is a coordinate of a vector representing a solution to a linear Mahler system.

Definition 2.2. A point $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is said to be *regular* with respect to the Mahler system (2.2) if the matrix $A(\boldsymbol{z})$ is well-defined and invertible at $T^k\boldsymbol{\alpha}$ for all nonnegative integers k.

Warning. Independently of the choice of the Mahler system (2.2), there are some unavoidable restrictions that one has to impose on the matrix transformation T and on the algebraic point $\boldsymbol{\alpha}$. When these conditions are fulfilled, the pair $(T, \boldsymbol{\alpha})$ is said to be *admissible.* We postpone the definition of an admissible pair to Section 4, but let us just already say that, in this respect, our results are as general as possible. With this formalism, all our results are concerned with values at some algebraic point $\boldsymbol{\alpha}$ of some Mahler functions $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ related by a system of the form (2.2) under the assumption that:

(a) the pair $(T, \boldsymbol{\alpha})$ is admissible,
(b) $\boldsymbol{\alpha}$ is regular with respect to (2.2).

As discussed in [1], these conditions are typical in Mahler's method.

2.2. **The lifting theorem.** As a first contribution, we prove the following result. Let us recall that a field extension $\mathbb{L}$ of a field $\mathbb{K}$ is said to be *regular*[3] if $\mathbb{K}$ is algebraically closed in $\mathbb{L}$ and $\mathbb{L}$ is separable over $\mathbb{K}$. If $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^n$, we let $\overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$ denote the algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ in $\overline{\mathbb{Q}}\{\boldsymbol{z} - \boldsymbol{\alpha}\}$[4].

Theorem 2.3 (Lifting). *Let $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be related by a system of functional equations of the form* (2.2)*. Let us assume that $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is a regular point with respect to* (2.2) *and that the pair $(T, \boldsymbol{\alpha})$ is admissible. Then for every homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that*

$$P(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) = 0\,,$$

there exists a homogeneous polynomial $Q \in \overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}[X_1, \ldots, X_m]$ such that

$$Q(\boldsymbol{z}, f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) = 0 \quad \text{and} \quad Q(\boldsymbol{\alpha}, X_1, \ldots, X_m) = P(X_1, \ldots, X_m).$$

Furthermore, if $\overline{\mathbb{Q}}(\boldsymbol{z})(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$, then there exists such a polynomial Q in $\overline{\mathbb{Q}}[\boldsymbol{z}, X_1, \ldots, X_m]$.

Theorem 2.3 is the first that applies to *all* linear Mahler systems in several variables, that is, without any restriction on the matrix $A(\boldsymbol{z})$. Furthermore, the quantitative Equality (2.1) is replaced by a qualitative statement: any algebraic relation over $\overline{\mathbb{Q}}$ between the values $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$ can be lifted to a similar algebraic relation over $\overline{\overline{\mathbb{Q}}(\boldsymbol{z})}$ between the functions $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$. Such a qualitative refinement is a key for applications.

Remark 2.4. Theorems 2.3 also applies to nonhomogeneous polynomials, for we can always turn an inhomogeneous relation into an homogeneous one by adding the constant function $f_0 \equiv 1$ to the system and replacing the matrix $A(\boldsymbol{z})$ by

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A(\boldsymbol{z}) \end{array}\right).$$

As a corollary of the lifting theorem, we deduce the following result.

[3]The reader will take care that we use two totally different notions of regularity in this paper.

[4]The ring $\overline{\mathbb{Q}}\{\boldsymbol{z} - \boldsymbol{\alpha}\}$ is the ring of convergent power series in $(\boldsymbol{z} - \boldsymbol{\alpha})$.

Corollary 2.5. *Let $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be related by a system of the form* (2.2). *Let us assume furthermore that $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is a regular point with respect to* (2.2) *and that the pair $(T,\boldsymbol{\alpha})$ is admissible. Then*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))\,. \tag{2.3}$$

Now, let us compare Theorem 1.1 with previous results on the subject.

The case $n = 1$. In that case, the operator T takes the simple form $z \mapsto z^q$, where $q \geq 2$ is an integer, and the pair (T,α) is admissible as soon as $0 < |\alpha| < 1$. Furthermore, the field extension $\overline{\mathbb{Q}}(z)(f_1(z),\ldots,f_m(z))$ is always regular. After several partial results due to Mahler, Kubota, and Loxton and van der Poorten, Ku. Nishioka [49] finally proved in 1990 that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha),\ldots,f_m(\alpha)) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\ldots,f_m(z)) \tag{2.4}$$

for all matrices $A(z)$ and all regular points $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. This is certainly a landmark result in Mahler's method. The proof of Nishioka's theorem is based on some technics from commutative algebra first introduced in the framework of algebraic independence by Nesterenko. More recently, Philippon [53] and then the authors [8] refine Nishioka's theorem by proving the case $n = 1$ of Theorem 2.3, which we refer to as Philippon's lifting theorem. Similar lifting theorems have first been obtained in the framework of linear differential equations (*e.g.*, Siegel E-functions) by Nesterenko and Shidlovskii [47], by Beukers [17] using some results of André [13, 14] on the theory of E-operators, and then by André [15]. A recent proof of Philippon's lifting theorem in the spirit of [15] is also given in [48]. In [53, 8, 48], the latter is derived from Nishioka's theorem, while our proof of Theorem 2.3 has little in common with these papers and [49]. In particular, it provides a new and more elementary way to prove the theorems of Nishioka and Philippon.

The case $n \geq 2$. Unfortunately, the method used for proving Nishioka's theorem hardly generalizes to higher dimension. In 1982, Loxton and van der Poorten [39] published a paper claiming that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})) \tag{2.5}$$

when the matrix $A(\boldsymbol{0})$ is well-defined and nonsingular, the pair $(T,\boldsymbol{\alpha})$ is admissible, and $\boldsymbol{\alpha}$ is a regular algebraic point. It was the main result published in this area, but unfortunately some argument in their proof is flawed. This is reported, for instance, by Nishioka in [49]. In the end, Mahler's method in several variables has been applied successfully only for the two following much more restricted classes of matrices. In 1977, Kubota [31] proved that Equality (2.5) holds true when the matrix $A(\boldsymbol{z})$ is *almost diagonal*, that is, when the functions $f_i(\boldsymbol{z})$ satisfy a system of equations of the form

$$\begin{pmatrix} 1 \\ f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0\cdots & & 0 \\ \hline b_1(\boldsymbol{z}) & a_1(\boldsymbol{z}) & & \\ \vdots & & \ddots & \\ b_m(\boldsymbol{z}) & & & a_m(\boldsymbol{z}) \end{array}\right) \begin{pmatrix} 1 \\ f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix} \tag{2.6}$$

where $a_i(\boldsymbol{z}), b_i(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z)$ have no pole at $\boldsymbol{0}$, and $a_i(\boldsymbol{0}) \neq 0$. Such systems are precisely those arising from the study of several inhomogeneous equations of order one. A variant of this result is due to Nishioka [51], who proved in 1996 that Equality (2.5) also holds true when the matrix $A(\boldsymbol{z})$ is *almost constant*, that is, for systems of the form

$$\begin{pmatrix} 1 \\ f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline b_1(\boldsymbol{z}) & & & \\ \vdots & & B & \\ b_m(\boldsymbol{z}) & & & \end{array}\right) \begin{pmatrix} 1 \\ f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix} \tag{2.7}$$

where $B \in \mathrm{GL}_m(\overline{\mathbb{Q}})$, and $b_i(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$ have no pole at $\mathbf{0}$. The proof of these results (and also of the failed attempt by Loxton and van der Poorten) follow closely the approach initiated by Mahler in [42]. We stress that, so far, this remained the only available strategy to tackle this problem (see Section 2.4).

2.3. **The two purity theorems.** According to the lifting theorem, the study of the algebraic relations between the values of Mahler functions related by a system of equations of the form (2.2) can be reduced to the easier study of the algebraic relations between the functions themselves. However, easier does not necessarily mean *easy*, and, so far, only the linear relations between M-functions have been fully understood [8, 9]. Our second main result is of a different nature. It states that, when evaluated at sufficiently independent algebraic points, Mahler functions associated with transformations having the same spectral radius *always* behave independently. The main feature of this result is that there is no need to check any kind of independence between the Mahler functions under consideration.

To state this result properly, we first need some notation. Let us consider several tuples of complex numbers

$$\mathcal{E}_1 = (\zeta_{1,1}, \ldots, \zeta_{1,s_1}), \ldots, \mathcal{E}_r = (\zeta_{r,1}, \ldots, \zeta_{r,s_r}) .$$

With every i, $1 \leq i \leq r$, we associate a vector of indeterminates $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,s_i})$, and we let

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i) := \left\{P(\boldsymbol{X}_i) \in \overline{\mathbb{Q}}[\boldsymbol{X}_i] : P(\zeta_{i,1}, \ldots, \zeta_{i,s_i}) = 0\right\}$$

denote the ideal of algebraic relations over $\overline{\mathbb{Q}}$ between the coordinates of $\mathcal{E}_i$. We also consider the tuple $\mathcal{E} = (\zeta_{1,1}, \ldots, \zeta_{r,s_r})$ obtained by concatenation of the tuples $\mathcal{E}_i$, and we set $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$ and

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) := \left\{P(\boldsymbol{X}) \in \overline{\mathbb{Q}}[\boldsymbol{X}] : P(\zeta_{1,1}, \ldots, \zeta_{r,s_r}) = 0\right\} .$$

We say that $P \in \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ is a *pure algebraic relation* with respect to $\mathcal{E}_i$ if it belongs to the extended ideal

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) := \mathrm{span}_{\overline{\mathbb{Q}}[\boldsymbol{X}]}\{P(\boldsymbol{X}_i) : P \in \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\} .$$

Our second main result reads as follows.

Theorem 2.6 (Purity–Independent points)**.** *Let $r \geq 2$ be an integer. For every integer i, $1 \leq i \leq r$, let us consider a linear Mahler system*

$$\begin{pmatrix} f_{i,1}(T_i\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(T_i\boldsymbol{z}_i) \end{pmatrix} = A_i(\boldsymbol{z}_i) \begin{pmatrix} f_{i,1}(\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(\boldsymbol{z}_i) \end{pmatrix} \tag{2.8.i}$$

where $A_i(\boldsymbol{z}_i)$ belongs to $\mathrm{GL}_{m_i}(\overline{\mathbb{Q}}(\boldsymbol{z}_i))$, $\boldsymbol{z}_i := (z_{i,1},\ldots,z_{i,n_i})$ is a tuple of indeterminates, and T_i is an $n_i\times n_i$ matrix with nonnegative integer coefficients and with spectral radius $\rho(T_i)$. Let $\boldsymbol{\alpha}_i = (\alpha_{i,1},\ldots,\alpha_{i,n_i}) \in (\overline{\mathbb{Q}}^\star)^{n_i}$, $\mathcal{E}_i$ be a subtuple of $(f_{i,1}(\boldsymbol{\alpha}_i),\ldots,f_{i,m_i}(\boldsymbol{\alpha}_i))$, and $\mathcal{E} = (\mathcal{E}_1,\ldots,\mathcal{E}_r)$. Suppose that the two following conditions hold.

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t.* (2.8.*i*) *and $(T_i,\boldsymbol{\alpha}_i)$ is admissible.*
(ii) *$\rho(T_1)=\cdots=\rho(T_r)$ and there is no nonzero tuple $(\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r)\in\mathbb{Z}^N$, $N=n_1+\cdots+n_r$, such that $(T_1^k\boldsymbol{\alpha}_1)^{\boldsymbol{\mu}_1}\cdots(T_r^k\boldsymbol{\alpha}_r)^{\boldsymbol{\mu}_r}=1$, for all k in an arithmetic progression.*

Then
$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^r \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})\,.$$

In other words, the only algebraic relations between the coordinates of $\mathcal{E}$ are those that can be trivially derived from the pure algebraic relations with respect to the coordinates of each $\mathcal{E}_i$.

The first results dealing with values of Mahler functions at independent points are due to Mahler [41] and are limited to linear independence over $\overline{\mathbb{Q}}$. Some generalization are due to Kubota [31] and to Loxton and van der Poorten [37]. All these results are restricted to the study of several inhomogeneous equations of order one.

Remark 2.7. Condition (ii) is clearly satisfied when all the algebraic numbers $\alpha_{1,1},\ldots,\alpha_{r,n_r}$ are multiplicatively independent.

Let us turn to our third main result. It states that values at algebraic points of Mahler functions associated with sufficiently independent transformations *always* behave independently. As with Theorem 2.6, the main advantage is that there is no need to check any kind of functional independence. Again, this result is expressed in terms of purity.

Theorem 2.8 (Purity–Independent transformations)**.** *We continue with the notation of Theorem 2.6. Suppose that the two following conditions hold.*

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t.* (2.8.*i*) *and $(T_i,\boldsymbol{\alpha}_i)$ is admissible.*
(ii) *The spectral radii $\rho(T_1),\ldots,\rho(T_r)$ are pairwise multiplicatively independent.*

Then
$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^r \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})\,.$$

In 1976, Kubota [30] and van der Poorten [54], first envisaged the possibility of extending Mahler's method in order to consider simultaneously several Mahler systems associated with independent transformations. In [30], Kubota gave a sketch of proof in a very specific case and announced a paper on this problem, but the latter never appeared in print. Then Loxton and van der Poorten [38] stated some related result, but the corresponding proof is incomplete (see [50, p. 89]). In 1987, van der Poorten [55] made this guess more ambitious and precise, pointing out several striking consequences that would follow from results he expected to prove in his collaboration with Loxton. However, these authors did not publish any new paper on this problem.

In the end, only examples limited to the study of several inhomogeneous equations of order one have been worked out by Nishioka [50] and Masser [45]. In contrast, Theorem 2.8 applies to arbitrary linear Mahler systems, and to a much larger class of transformation matrices and algebraic points.

Of course, Theorems 2.6 and 2.8 are strong statements about algebraic independence.

Corollary 2.9. *We continue with the assumptions of Theorems 2.6 or 2.8. The following equality holds true:*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,.$$

Remark 2.10. In geometric terms, Theorems 2.6 and 2.8 can be rephrased by saying that the affine $\overline{\mathbb{Q}}$-variety associated with the ideal $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ is isomorphic to the cartesian product of the affine $\overline{\mathbb{Q}}$-varieties associated with the ideals $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$, $1 \leq i \leq r$. Indeed, we prove that their coordinate rings are isomorphic. That is,

$$\frac{\overline{\mathbb{Q}}[\boldsymbol{X}]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})} \cong \frac{\overline{\mathbb{Q}}[\boldsymbol{X}_1]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_1)} \otimes_{\overline{\mathbb{Q}}} \cdots \otimes_{\overline{\mathbb{Q}}} \frac{\overline{\mathbb{Q}}[\boldsymbol{X}_r]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_r)}\,.$$

2.4. **Main new ingredients.** As already mentioned, all previous results concerning the transcendence theory of linear Mahler systems in several variables are very much inspired by the early work of Mahler [42]. We also start with the same initial strategy, but we add a number of fundamental new ingredients, including Hilbert's Nullstellensatz, tools from ergodic Ramsey theory, and a new vanishing theorem.

In all previous works, a crucial step consists in expressing the coordinates of the iterated matrix $A_k(\boldsymbol{z}) := A(\boldsymbol{z})A(T\boldsymbol{z})\cdots A(T^{k-1}\boldsymbol{z})$ associated with the Mahler system (2.2) in terms of linear combination some convergent power series of the form $g_i(T^k\boldsymbol{z})$, possibly twisted by some multivariate exponential polynomials. This is really of great importance for one can then apply some vanishing theorems to the power series $g_i(\boldsymbol{z})$. This step has gradually become more difficult in the aforementioned works, as the matrices under consideration have taken a more general form. Its complexity culminated in [10]. Unfortunately, one cannot expect to find this kind of expression when $A(\boldsymbol{z})$ is not regular singular in the sense of [10, Definition 1.1]. Hence this strategy suffers from an intrinsic limitation, which prevents from dealing with arbitrary Mahler systems. We overcome this main deficiency by defining the so-called *relation matrices* in Section 7. Their existence and main properties are obtained by means of Hibert's Nullstellensatz and the notion of piecewise syndetic set. Introducing these matrices is a cornerstone of the present work and the main novelty with respect to our two unpublished preprints [10, 11].

In order to apply our results to transformations T and points $\boldsymbol{\alpha}$ that are as general as possible, it is of great importance to prove suitable *vanishing theorems.* That is, results that guarantee the nonvanishing of arbitrary multivariate analytic functions at special sets of points (typically, certain subsets of $\{T^k\boldsymbol{\alpha},\, k \geq 0\}$). In the case of a single transformation, Masser [44] solved this problem in a rather definitive way. We note that Masser's vanishing theorem (in fact a refinement using the notion of piecewise syndetic set) is already strong enough to prove Theorems 2.3 and 2.6. In fact, we only need the identity theorem for reproving Nishioka's theorem and Philippon's lifting theorem. Unfortunately, Masser's vanishing theorem is not suited to deal with Mahler systems associated with independent transformations. First results towards this goal were proved by Nishioka [50] and, again, by Masser [45]. Unfortunately, they remain too restricted for proving Theorem 2.8. In 2005, Corvaja and Zannier [22, Theorem 3] deduced from the subspace theorem a general result concerning the vanishing at $\mathcal{S}$-units of analytic multivariate power series with algebraic coefficients. They already noticed that it could be relevant for Mahler's method. Using the flexibility of their result and the notion of piecewise syndetic set, we cook up in Section 5 our own vanishing theorem, which is specifically shaped for our purpose.

2.5. **Relevance of Mahler's formalism.** To end this section, we recall two major advantages that this multivariate formalism offers.

First, adding variables makes it possible to deal with values of M-functions at *different algebraic points*. Let us give a basic example. With the function $\mathfrak{f}(z) = \sum_{n=0}^{\infty} z^{2^n}$, we can associate the two variables linear T-Mahler system

$$\begin{pmatrix} 1 \\ \mathfrak{f}(z_1^2) \\ \mathfrak{f}(z_2^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -z_1 & 1 & 0 \\ -z_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{f}(z_1) \\ \mathfrak{f}(z_2) \end{pmatrix}, \quad \text{where } T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{2.9}$$

By Theorem 4.6, the point $\boldsymbol{\alpha} := (1/2, 1/3)$ is regular w.r.t. (2.9) and the pair $(T, \boldsymbol{\alpha})$ is admissible. The key point is that the transcendence of $\mathfrak{f}(z)$ gives *for free* the algebraic independence over $\overline{\mathbb{Q}}(z_1, z_2)$ of the functions $\mathfrak{f}(z_1)$ and $\mathfrak{f}(z_2)$. By Corollary 2.5, it follows that $\mathfrak{f}(1/2)$ and $\mathfrak{f}(1/3)$ are algebraically independent over $\overline{\mathbb{Q}}$. This important principle really takes shape, and acquires great generality, with Theorems 1.1 and 2.6.

The second advantage of Mahler's multivariate formalism comes from the possibility of dealing with a much larger class of one-variable functions obtained by suitable specializations of Mahler functions in several variables. Mahler's favorite example is the family of the Hecke-Mahler functions

$$\mathfrak{f}_\omega(z) = \sum_{n=0}^{\infty} \lfloor n\omega \rfloor z^n ,$$

where ω is a quadratic irrational real number. Though $\mathfrak{f}_\omega(z)$ is not an M-function[5], we have that $\mathfrak{f}_\omega(z) = F_\omega(z, 1)$, where

$$F_\omega(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\lfloor n_1\omega \rfloor} z_1^{n_1} z_2^{n_2}$$

is a Mahler function in two variables. In another direction, Cobham [20] proved that generating functions of morphic sequences are specializations of the form $F(z, \ldots, z)$ for some multivariate Mahler functions $F(z_1, \ldots, z_n)$. Some related applications of our main results can be found in [11].

[5] This fact only very recently became known. It is an easy consequence of Theorem 2.8, but it can also be obtained by combining the results in [3] and [23].

3. Notation

We fix here some notation that we will use all along this paper. Let d be a positive integer and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{C} \setminus \{0\})^d$. If $T = (t_{i,j})_{1 \le i,j \le d}$ is a $d \times d$ matrix with nonnegative integer coefficients, we let $\rho(T)$ denote its spectral radius. Furthermore, we recall that

$$T\boldsymbol{\alpha} = (\alpha_1^{t_{1,1}}\alpha_2^{t_{1,2}}\cdots\alpha_d^{t_{1,d}}, \ldots, \alpha_1^{t_{d,1}}\alpha_2^{t_{d,2}}\cdots\alpha_d^{t_{d,d}})$$

and that we let $T(\boldsymbol{x})$ denote the usual matrix product between T and a column vector $\boldsymbol{x} \in \mathbb{C}^d$. If $T_1, \ldots, T_d$ are matrices, we let $T_1 \oplus \cdots \oplus T_d$ denote the direct sum of these matrices. That is,

$$T_1 \oplus \cdots \oplus T_d = \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_d \end{pmatrix}.$$

Let $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, then $\boldsymbol{\alpha}^{\boldsymbol{k}}$ stands for $\alpha_1^{k_1}\cdots\alpha_d^{k_d}$, so that $(T\boldsymbol{\alpha})^{\boldsymbol{k}} = \boldsymbol{\alpha}^{\boldsymbol{k}T}$. Given a d-tuple of natural numbers $\boldsymbol{k} = (k_1, \ldots, k_d)$, we set $|\boldsymbol{k}| = k_1 + \cdots + k_d$. We use the same symbol $\|\cdot\|$ to denote both the maximum norm of $\mathbb{C}^d$ and the maximum norm of $\mathcal{M}_d(\mathbb{C})$. We let $|\xi|$ denote the module of the complex number ξ.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$. We define a partial order on $\mathbb{N}^n$ by setting $\boldsymbol{\lambda} \le \boldsymbol{\gamma}$ if $\lambda_i \le \gamma_i$, $1 \le i \le d$. We also set

$$\binom{\boldsymbol{\lambda}}{\boldsymbol{\gamma}} = \prod_{i=1}^{d} \binom{\lambda_i}{\gamma_i},$$

the product of binomial coefficients associated with each coordinate of $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. Given a positive integer h, a matrix $M = (m_{i,j})_{1 \le i,j \le h}$ with coefficients in some ring, and a matrix $\boldsymbol{\mu} = (\mu_{i,j})_{1 \le i,j \le h}$ with nonnegative integer coefficients, we set

$$M^{\boldsymbol{\mu}} = \prod_{1 \le i,j, \le h} m_{i,j}^{\mu_{i,j}}.$$

We use the standard Landau notation $\mathcal{O}$. We also use the notation $\gg$ as follows. Writing that some property holds true for all integers $\lambda_1 \gg \lambda_2, \lambda_3$ means that the corresponding property holds true for all λ_1 that is sufficiently large w.r.t. λ_2 and λ_3, while writing that some property holds true for all integers $\lambda_1 \gg \lambda_2 \gg \lambda_3$ means that the corresponding property holds true for all λ_1 that is sufficiently large w.r.t. λ_2, assuming that λ_2 is itself sufficiently large w.r.t. λ_3.

Given a positive real number R and $\boldsymbol{\alpha} \in \mathbb{C}^d$, we let

$$\mathcal{D}(\boldsymbol{\alpha}, R) = \{\boldsymbol{\theta} \in \mathbb{C}^d \ : \ \|\boldsymbol{\theta} - \boldsymbol{\alpha}\| < R\}$$

denote the open polydisc of center $\boldsymbol{\alpha}$ and radius R. By definition, an element $g \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ has a unique expansion of the form

$$g(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^d} g_{\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}},$$

which converges in some neighborhood of the origin. The *radius of convergence* of g is defined as the supremum of the positive real numbers R such that the power series defining g is convergent on $\mathcal{D}(\mathbf{0}, R)$. By [19, Proposition 2.2], when the radius of convergence of g is finite and equal to R_0, the power series defining g is absolutely convergent on $\mathcal{D}(\mathbf{0}, R_0)$. By specialization, we deduce from the Cauchy-Hadamard theorem that if $g(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^d} g_{\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$, then

$$|g_{\boldsymbol{\lambda}}| = \mathcal{O}\left(R^{-|\boldsymbol{\lambda}|}\right) , \tag{3.1}$$

for all positive real numbers R smaller than the radius of convergence of g.

We let $H(\cdot)$ denote the *absolute Weil height* over the projective space $\mathbb{P}^d(\overline{\mathbb{Q}})$. Given $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \overline{\mathbb{Q}}^d$, we also write $H(\boldsymbol{\beta})$ instead of $H(\beta_1 : \cdots : \beta_d : 1)$. We will only use basic properties of the Weil height and we refer the interested reader to [60, Chapter 3] for more details.

4. Admissibility conditions

A well-known feature of Mahler's method is that, independently of the choice of the matrix $A(\boldsymbol{z})$ defining the system (2.2), some unavoidable restrictions on the transformation T and on the point $\boldsymbol{\alpha}$ are required.

Definition 4.1. Let T be an $n \times n$ matrix with nonnegative integer coefficients and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$. The pair $(T, \boldsymbol{\alpha})$ is said to be *admissible* if there exist two real numbers $\rho > 1$ and $c > 0$ such that the following three conditions hold.

(a) The coefficients of the matrix T^k belong to $\mathcal{O}(\rho^k)$.
(b) Set $T^k \boldsymbol{\alpha} := (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$. Then $\log |\alpha_i^{(k)}| \leq -c\rho^k$, for every integer i, $1 \leq i \leq n$, and all sufficiently large integers k.
(c) If $f(\boldsymbol{z}) \in \mathbb{C}\{\boldsymbol{z}\}$ is nonzero, then there are infinitely many integers k such that $f(T^k \boldsymbol{\alpha}) \neq 0$.

The strength of our results strongly depends on our ability to provide simple and natural conditions that imply Conditions (a), (b), and (c). The latter are in fact necessary to apply Mahler's method (see [40]). Though they appear naturally in proofs, it is not that easy, at first glance, to see how to check them. We provide here a simple characterization of matrices and algebraic points satisfying these conditions, gathering results of Kubota [31], Loxton and van der Poorten [36, 37], and mainly Masser [44].

Definition 4.2. Let T be an $n \times n$ matrix with nonnegative integer coefficients and with spectral radius $\rho(T)$. We say that T belongs to the class $\mathcal{M}$ if it satisfies the following three conditions.

(i) It is nonsingular.
(ii) None of its eigenvalues are roots of unity.
(iii) There exists an eigenvector with positive coordinates associated with the eigenvalue $\rho(T)$.

In particular, if T belongs to the class $\mathcal{M}$, then $\rho(T) > 1$.

Remark 4.3. Let us consider r Mahler systems associated with transformations $T_1, \ldots, T_r$ in $\mathcal{M}$, all having the same spectral radius ρ. Then the direct sum $T_1 \oplus \cdots \oplus T_r$ also belongs to the class $\mathcal{M}$. More generally, given r Mahler systems, associated with transformation matrices $T_1, \ldots, T_r \in \mathcal{M}$ with pairwise multiplicatively dependent spectral radius, it is possible to gather them into a larger Mahler system whose transformation matrix also belongs to the class $\mathcal{M}$, and then to apply Theorem 2.3.

Given a one-variable Mahler system associated with a matrix $A(z)$, we can consider the same system twice but with different variables. That is, the system associated with the matrix

$$\begin{pmatrix} A(z_1) & 0 \\ 0 & A(z_2) \end{pmatrix} .$$

This shows that some kind of minimal independence between the coordinates of the point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is required in order to apply Mahler's method. Typically, we cannot consider a point of the form (α, α) in that case.

Definition 4.4. A point $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is said to be *T-independent* if there is no nonzero n-tuple of integers $\boldsymbol{\mu}$ for which $(T^k\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 1$ for all k in an arithmetic progression.

Remark 4.5. According to Definition 4.4, Condition (ii) of Theorem 2.6 is equivalent to the fact that the point $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$ is T-independent with respect to the direct sum $T = T_1 \oplus \cdots \oplus T_r$.

With these definitions, we have the following characterization of admissibility, which makes our main results very convenient to apply.

Theorem 4.6. *Let T be an $n \times n$ matrix with nonnegative integer coefficients and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$. Then the pair $(T, \boldsymbol{\alpha})$ is admissible if and only if T belongs to the class $\mathcal{M}$, $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = \mathbf{0}$, and $\boldsymbol{\alpha}$ is T-independent.*

Remark 4.7. There is no difficulty in checking whether or not a matrix belongs to the class $\mathcal{M}$. Furthermore, if $\alpha_1, \ldots, \alpha_n \in \mathbb{C} \setminus \{0\}$ are multiplicatively independent complex numbers, then $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is *a fortiori* T-independent.

Proof of Theorem 4.6. We first prove the reverse direction. Let us assume that T belongs to $\mathcal{M}$, $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is T-independent, and that $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = 0$. Then there exists k_0 such that $\|T^{k_0}\boldsymbol{\alpha}\| < 1$. We observe that if the pair $(T, T^{k_0}\boldsymbol{\alpha})$ is admissible, then so is the pair $(T, \boldsymbol{\alpha})$. Thus, we can assume without any loss of generality that $\|\boldsymbol{\alpha}\| < 1$. We let $\boldsymbol{x}$ denote the transpose of the vector

$$(-\log|\alpha_1|, \ldots, -\log|\alpha_n|) ,$$

whose coordinates are all positive. By assumption, T has a positive eigenvector associated with the eigenvalue $\rho(T)$. Let us choose such an eigenvector $\boldsymbol{\mu}$ whose coordinates are all smaller than those of $\boldsymbol{x}$. We have

$$-\log\|T^k\boldsymbol{\alpha}\| = \|T^k(\boldsymbol{x})\| = \|T^k(\boldsymbol{\mu}) + T^k(\boldsymbol{x}-\boldsymbol{\mu})\| > \|T^k(\boldsymbol{\mu})\| = \rho(T)^k\|\boldsymbol{\mu}\| ,$$

for all $k \in \mathbb{N}$, because $T^k(\boldsymbol{x}-\boldsymbol{\mu})$ has positive coordinates. Condition (b) is thus satisfied with $\rho = \rho(T)$. Following Loxton and van der Poorten [36], Condition (a) is satisfied for the matrix T belongs to $\mathcal{M}$. Finally, Masser vanishing theorem [44] implies that Condition (c) holds since $\boldsymbol{\alpha}$ is T-independent. Hence, the pair $(T, \boldsymbol{\alpha})$ is admissible.

Now, we prove the forward direction. Let $(T, \boldsymbol{\alpha})$ be an admissible pair. We first note that Condition (b) implies that $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = 0$.

Replacing $\boldsymbol{\alpha}$ by $T^k\boldsymbol{\alpha}$ for some k if necessary, we can thus assume without any loss of generality that $\|\boldsymbol{\alpha}\| < 1$. By [31], Condition (c) implies that the matrix T is nonsingular and that none of its eigenvalues is a root of unity. Hence its spectral radius $\rho(T)$ is larger than 1. Let us prove that there exists an eigenvector with positive coordinates associated with the eigenvalue $\rho(T)$. Let ρ be as in Definition 4.1. We first infer from Conditions (a) and (b) that $\rho = \rho(T)$. Since the coefficients of T are nonnegative integers, for every eigenvalue ρ' with $|\rho'| = \rho$, there is a root of unity μ such that $\rho' = \mu\rho$ (see, for instance, [28, Theorem 2, p. 65 & Chapter III §4]). Replacing T by some power if necessary, we can assume that ρ is larger than every other eigenvalue of T. Let E_ρ denote the eigenspace associated with ρ. Condition (a) implies that the characteristic space associated with ρ is equal to E_ρ, since otherwise the sequence T^k/ρ^k would not be bounded. Hence E_ρ has a T-invariant complement, say $E_\rho^\perp$. From Condition (b), we infer the existence of a positive real number γ such that every coordinate of $T^k(\boldsymbol{x})$ is larger than $\gamma\rho^k$. Set

$$\boldsymbol{x} = \boldsymbol{e} + \boldsymbol{e}^\perp, \text{ where } \boldsymbol{e} \in E_\rho \text{ and } \boldsymbol{e}^\perp \in E_\rho^\perp .$$

Suppose that there is no vector in E_ρ with positive coordinates. Then, for some j, the jth coordinate of $\boldsymbol{e}$ is nonpositive. Since $T^k(\boldsymbol{x}) = \rho^k\boldsymbol{e} + T^k(\boldsymbol{e}^\perp)$, we deduce that the jth coordinate of $T^k(\boldsymbol{e}^\perp)$ is larger than $\gamma\rho^k$. Since all eigenvalues of T on $E_\rho^\perp$ are smaller than ρ, we obtain a contradiction. This shows that T belongs to the class $\mathcal{M}$.

Finally, $\boldsymbol{\alpha}$ is T-independent. Indeed, otherwise there would exist two tuples of nonnegative integers $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$, not both zero, such that $P(T^k\boldsymbol{\alpha}) = 0$ for infinitely many k, where $P(\boldsymbol{z}) = z_1^{s_1}\cdots z_n^{s_n}$ - $z_1^{t_1}\cdots z_n^{t_n}$, providing a contradiction with Condition (c). □

5. A new vanishing theorem

As already mentioned, it is of great importance to find natural conditions that ensure nonvanishing properties similar to Condition (c) in Definition 4.1. Of course, our goal is to obtain a *vanishing theorem* that can be applied to transformation matrices and points which are as general as possible. Our contribution to this problem is Theorem 5.4.

In the framework of Mahler's method, several vanishing theorems have been formulated by saying that a nonzero multivariate power series cannot vanish at all points in some well-structured large sets. The latter are obtained by iteration of the transformation matrix and usually involve arithmetic progressions. In order to prove our main theorems, we need to replace these *well-structured sets* by sets which remain large but offer much more flexibility. We use the notion of piecewise syndetic set, which is classical in Ramsey theory, especially in its ergodic counterpart. As we just said, it can be though of as a notion of largeness for subsets of $\mathbb{N}$. Furthermore, Brown's lemma (see (ii) in Lemma 5.2) shows that such sets are partition regular, and thus much more robust in terms of partitions than arithmetic progressions.

Definition 5.1. A set $\mathcal{Z} \subset \mathbb{N}$ is said to be *piecewise syndetic* if there exists a natural number $B \geq 1$ such that for any given integer $C \geq 2$ there exist

$l_1 < \cdots < l_C$ in $\mathcal{Z}$ such that

$$l_{i+1} - l_i \leq B, \qquad 1 \leq i < C\,.$$

In this case, we say that B is a *bound* for $\mathcal{Z}$. A set $\mathcal{Z} \subset \mathbb{N}$ is said to be *negligible* if it is not piecewise syndetic, while it is said to be *full* if its complement is negligible.

Let us recall that a subset of $\mathbb{N}$ is said to be *syndetic*, or sometimes *relatively dense*, if it has bounded gaps. A subset of $\mathbb{N}$ is said to be *thick* if it contains arbitrarily long intervals. Thus piecewise syndetic sets are those that can be obtained as the intersection of a syndetic set and a thick set. In the rest of this section, as well as all along Section 7, we will use heavily the following results.

Lemma 5.2. *Let $\mathcal{Z} \subset \mathbb{N}$ be a piecewise syndetic set with bound B. Then the following properties hold.*

(i) *If $\mathcal{Z} \subset \mathcal{Z}' \subset \mathbb{N}$, then $\mathcal{Z}'$ is also piecewise syndetic.*
(ii) *If $\mathcal{Z} \subset \cup_{i=1}^{s} \mathcal{Z}_i$, then at least one of the sets $\mathcal{Z}_i$ is piecewise syndetic.*
(iii) *Let l_0 be a natural number. The set*

$$\mathcal{Z}_0 := \{l \in \mathcal{Z} : (l + \{l_0, \ldots, l_0 + B\}) \cap \mathcal{Z} \neq \emptyset\}$$

is piecewise syndetic.
(iv) *The set $\mathcal{Z}$ contains arbitrarily long arithmetic progressions.*

Proof. The point (i) immediately follows from the definition, while points (ii) and (iv) correspond to classical results respectively known as Brown's lemma (see [18]) and Szemerédi's theorem [59]. Let us prove (iii). Let l_0 and C be two natural numbers and let a be the smallest integer such that $aB \geq l_0$. Since $\mathcal{Z}$ is piecewise syndetic, there exists a sequence $l_1 < l_2 < \cdots < l_{C+aB}$ of integers in $\mathcal{Z}$ such that $l_{i+1} - l_i < B$. Let $i \in \{1, \ldots, C\}$. Then, $l_i + l_0 \leq l_i + aB \leq l_{i+aB}$. There thus exists an integer $j \leq aB$ such that $l_i + l_0 \leq l_{i+j} \leq l_i + l_0 + B$. Hence $l_i \in \mathcal{Z}_0$. Thus, $l_1, \ldots, l_C$ all belong to the set $\mathcal{Z}_0$, which proves that this set is piecewise syndetic. $\square$

Part of Lemma 5.2 can be naturally rephrased as follows.

Lemma 5.3. *The following properties hold.*

(i) *A finite union of negligible sets is negligible.*
(ii) *A finite intersection of full sets is full.*
(iii) *If $\mathcal{Z}_1$ is full and $\mathcal{Z}_2$ is negligible, then $\mathcal{Z}_1 \setminus \mathcal{Z}_2$ is full.*

Proof. Point (i) follows directly from Point (ii) of Lemma 5.2. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ be full sets. By (i), we obtain that $(\cap \mathcal{Z}_i)^c = \cup(\mathcal{Z}_i^c)$ is negligible, which proves (ii). Let us prove (iii). By assumption, $\mathcal{Z}_1^c$ is negligible. By (i), the set $\mathcal{Z}_2 \cup \mathcal{Z}_1^c$ is also negligible. Since $\mathcal{Z}_2 \cup \mathcal{Z}_1^c = (\mathcal{Z}_1 \setminus \mathcal{Z}_2)^c$, we obtain that $\mathcal{Z}_1 \setminus \mathcal{Z}_2$ is full, as wanted. $\square$

We are now ready to state our vanishing theorem.

Theorem 5.4. *Let $T_1, \ldots, T_r$ be matrices in the class $\mathcal{M}$ whose spectral radii $\rho(T_1), \ldots, \rho(T_r)$ are pairwise multiplicatively independent. Let n_i denote the* *size of the matrix* T_i *and set* $N := \sum_{i=1}^r n_i$. *Set*

$$\Theta := \left(\frac{1}{\log \rho(T_1)}, \dots, \frac{1}{\log \rho(T_r)} \right) . \tag{5.1}$$

For every $l \in \mathbb{N}$, *we let* $\boldsymbol{k}_l := (k_{l,1}, \dots, k_{l,r})$ *denote a* r*-tuple of positive integers. Let us assume that*

$$\|\boldsymbol{k}_l - l\Theta\| = \mathcal{O}(1) . \tag{5.2}$$

Let $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_r) \in (\overline{\mathbb{Q}}^\star)^N$ *be such that the pair* $(T_i, \boldsymbol{\alpha}_i)$ *is admissible for every* i, *and let* $g(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ *be nonzero. Then the set*

$$\left\{ l \in \mathbb{N} : g(T_1^{k_{l,1}} \boldsymbol{\alpha}_1, \dots, T_r^{k_{l,r}} \boldsymbol{\alpha}_r) = 0 \right\}$$

is negligible.

Applying Mahler's method to several Mahler systems requires some uniform speed of convergence to the origin for the orbits of each algebraic point $\boldsymbol{\alpha}_i$ under the matrix transformations T_i. As noticed by van der Poorten [55], one way to overcome this difficulty is to iterate each transformation T_i k_i-times, and to choose the iteration vector $\boldsymbol{k} = (k_1, \dots, k_r)$ so that asymptotically the matrices $T_i^{k_i}$ have essentially the same radius of convergence. As explained by Lemma 5.5, this forces us to consider only iteration vectors $\boldsymbol{k}$ that remain at bounded distance from the real line $\mathbb{R}\Theta$, where Θ is defined by (5.1). This explains why the assumption (5.2) is natural in this framework. In the rest of this section, we set

$$T_{\boldsymbol{k}} := T_1^{k_1} \oplus \cdots \oplus T_r^{k_r} \text{ and } T_{\boldsymbol{k}}\boldsymbol{\alpha} := (T_1^{k_1}\boldsymbol{\alpha}_1, \dots, T_r^{k_r}\boldsymbol{\alpha}_r).$$

Lemma 5.5. *Let* $T_1, \dots, T_r$, $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_r$ *be as in Theorem 5.4 and let* $(\boldsymbol{k}_l)_{l \in \mathbb{N}}$ *be an arbitrary sequence with values in* $\mathbb{N}^r$. *Then the following properties are equivalent.*

- *There exist a real numbers* $\rho > 1$ *and* $c > 0$ *such that*

  $\|T_{\boldsymbol{k}_l}\| = \mathcal{O}(\rho^l)$ *and* $\log \|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\| \leq -c\rho^l$ *for all large enough* l.
- *There exists a real number* λ *such that* $\|\boldsymbol{k}_l - \lambda l \Theta\| = \mathcal{O}(1)$, *where* Θ *is defined as in* (5.1). *In that case, one has* $\rho = e^\lambda$. *In particular, if* $\lambda = 1$, *then* $\rho = e$.

Proof. Let us assume that there exists a real number λ such that $\|\boldsymbol{k}_l - \lambda l\Theta\| = \mathcal{O}(1)$. There thus exists a positive real number B, such that, for every $l \in \mathbb{N}$ and i, $1 \leq i \leq r$, we have $k_{i,l} = \lambda l / \log(\rho(T_i)) + \epsilon(i,l)$, for some real number $\epsilon(i,l)$ with $|\epsilon(i,l)| \leq B$. By Conditions (a) and (b) in Definition 4.1, we have, on the one hand, that

$$\left\| T_i^{k_{i,l}} \right\| = \mathcal{O}\left(\rho(T_i)^{k_{i,l}}\right) = \mathcal{O}\left(\rho(T_i)^{\lambda l / \log(\rho(T_i))}\right) = \mathcal{O}\left(e^{\lambda l}\right) ,$$

while, on the other hand,

$$\log \left\| T_i^{k_{i,l}} \boldsymbol{\alpha}_i \right\| \leq -c_i \rho(T_i)^{k_{i,l}} \leq -c_i' e^{\lambda l} ,$$

for all large enough l, where c_i and c_i' are positive real numbers. Setting $c := \min_i\{c_1', \dots, c_r'\}$, we obtain the desired estimate.

Now, let us assume that

$$\|T_{\boldsymbol{k}_l}\| = \mathcal{O}(\rho^l) \text{ and } \log \|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\| \leq -c\rho^l \text{ for all large enough } l.$$

Set $\lambda := \log(\rho)$ and let i be an integer with $1 \leq i \leq r$. Since the pair $(T_i, \boldsymbol{\alpha}_i)$ satisfies Conditions (a) and (b) of Definition 4.1, we have

$$\left\|T_i^{k_{i,l}}\right\| = \mathcal{O}\left(\rho(T_i)^{k_{i,l}}\right) \text{ and } \log \left\|T_i^{k_{i,l}}\boldsymbol{\alpha}_i\right\| \leq -c_i\rho(T_i)^{k_{i,l}}\,,$$

for some positive real number c_i and all large enough l. There thus exist two positive real number κ_i and γ_i such that

$$\kappa_i e^{\lambda l} \leq \rho(T_i)^{k_{i,l}} \leq \gamma_i e^{\lambda l}\,,$$

for all $l \in \mathbb{N}$. Taking the logarithm, it follows that

$$\log(\kappa_i) + \lambda l \leq \log(\rho(T_i))k_{i,l} \leq \log(\gamma_i) + \lambda l\,.$$

Dividing by $\log(\rho(T_i))$, we see that $k_{i,l}$ remains at bounded distance from $\lambda l/\log(\rho(T_i))$. This ends the proof. □

Before proving Theorem 5.4, we need the two following auxiliary results. The proof of Theorem 5.4 is based on a vanishing theorem due to Corvaja and Zannier [22, Theorem 3]. The latter states that if the set of zeros of a multivariate analytic function with algebraic coefficients contains an infinite sequence of $\mathcal{S}$-unit points whose height does not grow too fast, then these points all belong to a finite number of translates of tori. The goal of the following two lemmas is to show that most points of the form $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$, $l \in \mathbb{N}$, avoid these tori.

Lemma 5.6. *We continue with the assumptions of Theorem 5.4. Then, for every nonzero integer N-tuple $\boldsymbol{\mu}$, the set*

$$\mathcal{Z} := \{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 1\}$$

is negligible.

Proof. We argue by contradiction, assuming that $\mathcal{Z}$ is piecewise syndetic. For every pair of nonnegative integers (l, m), with $m > 0$, we define the r-tuple of natural numbers $\boldsymbol{e} := \boldsymbol{e}(l, m)$ by

$$\boldsymbol{e} = \boldsymbol{k}_{l+m} - \boldsymbol{k}_l \tag{5.3}$$

and we set $\mathcal{E} := \{\boldsymbol{e}(l, m) :\ l,\, m \in \mathbb{N}\}$. Since by (5.2) we have $\boldsymbol{k}_l = l\Theta + \mathcal{O}(1)$, we obtain that

$$\boldsymbol{e}(l, m) = m\Theta + \mathcal{O}(1)\,, \tag{5.4}$$

which shows that $\mathcal{E}$ is infinite. However, given any positive integer m_0, the set $\{\boldsymbol{e}(l, m_0) :\ l \in \mathbb{N}\}$ is finite.

Let us remark that given any pair (β_1, β_2) of nonzero complex numbers that are not roots of unity, and any pair of natural numbers (i, j), $1 \leq i < j \leq r$, the set

$$\mathcal{E}_1 := \left\{\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathcal{E} : \beta_1^{e_i} = \beta_2^{e_j}\right\}$$

is finite. Indeed, the set of natural numbers u such that there exists a natural number v for which $\beta_1^u = \beta_2^v$ is an ideal of $\mathbb{Z}$. Let $u_0 \geq 0$ be a generator of this ideal and let $v_0 \in \mathbb{N}$ be such that $\beta_1^{u_0} = \beta_2^{v_0}$. For every $(e_1, \ldots, e_r) \in \mathcal{E}_1$, there exists an integer a such that $e_i = au_0$. We obtain that

$$\beta_2^{e_j} = \beta_1^{e_i} = \beta_1^{au_0} = \beta_2^{av_0} .$$

Since β_2 is nonzero and is not a root of unity, we have $e_j = av_0$. Hence $e_i/e_j = u_0/v_0 \in \mathbb{Q}$. Since $e_i = k_{i,l+m} - k_{i,l}$ and $e_j = k_{j,l+m} - k_{j,l}$, we get that

$$\frac{k_{j,l+m} - k_{j,l}}{k_{i,l+m} - k_{i,l}} = \frac{u_0}{v_0} \in \mathbb{Q} .$$

Now let us assume by contradiction that $\mathcal{E}_1$ is infinite. Then, there exist arbitrarily large integers m with this property. Letting m tends to infinity, we deduce from (5.4) that the ratio $\log \rho(T_i)/\log \rho(T_j)$ is rational. This provides a contradiction since by assumption $\rho(T_i)$ and $\rho(T_j)$ are multiplicatively independent. Hence $\mathcal{E}_1$ is finite.

Let us recall that, by assumption, none of the eigenvalues of the matrices T_i is equal to zero or to a root of unity. The previous reasoning shows that there exists a positive integer m_0 such that, for every $m \geq m_0$, every $l \in \mathbb{N}$, every eigenvalue λ_i of T_i, and every eigenvalue λ_j of T_j, $i \neq j$, we have

$$\lambda_i^{e_i} \neq \lambda_j^{e_j} , \tag{5.5}$$

where $\boldsymbol{e} = \boldsymbol{e}(l, m) = (e_1, \ldots, e_r)$. For such a vector $\boldsymbol{e}$, set

$$T_{\boldsymbol{e}} := T_1^{e_1} \oplus \cdots \oplus T_r^{e_r} .$$

Given a vector space $V \subset \mathbb{C}^N$, we have

$$V \subset \bigoplus_{i=1}^{r} \iota_i \circ \pi_i(V) ,$$

where we let $\pi_i : \mathbb{C}^N = \mathbb{C}^{n_1+\cdots+n_r} \to \mathbb{C}^{n_i}$ denote the projection defined by $\pi_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r) = \boldsymbol{x}_i$ and $\iota_i : \mathbb{C}^{n_i} \to \mathbb{C}^N$ be such that $\pi_i \circ \iota_i$ is the identity on $\mathbb{C}^{n_i}$. By (5.5), if V is invariant under $T_{\boldsymbol{e}}$, then

$$V = \bigoplus_{i=1}^{r} \iota_i \circ \pi_i(V) . \tag{5.6}$$

We are now ready to proceed with the proof of the lemma. Let us consider the column vector $\boldsymbol{x}$ whose transpose is the vector

$$(\log \alpha_{1,1}, \log \alpha_{1,2}, \ldots, \log \alpha_{1,n_1}, \log \alpha_{2,1}, \ldots, \log \alpha_{r,n_r}) ,$$

where log stands for a suitable determination of the logarithm (that is, the corresponding branch cut avoids all the coordinates of the points $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$, $l \in \mathbb{N}$). By assumption, we have

$$\langle \boldsymbol{\mu} \, , \, T_{\boldsymbol{k}_l}(\boldsymbol{x}) \rangle = 0$$

for all $l \in \mathcal{Z}$, where we let $\langle \, , \rangle$ denote the usual scalar product. Let U denote the orthogonal complement to the vector $\boldsymbol{\mu}$ in $\mathbb{C}^N$. This is a proper subspace of $\mathbb{C}^N$ defined over $\mathbb{Q}$, which contains all vectors $T_{\boldsymbol{k}_l}(\boldsymbol{x})$, $l \in \mathcal{Z}$. Given $\mathcal{Z}' \subset \mathcal{Z}$, we let $U(\mathcal{Z}')$ denote the smallest vector subspace of $\mathbb{C}^N$ defined over $\mathbb{Q}$ and containing all $T_{\boldsymbol{k}_l}(\boldsymbol{x})$, $l \in \mathcal{Z}'$. It follows that $U(\mathcal{Z}) \subset U$. Furthermore, if $\mathcal{Z}'' \subset \mathcal{Z}'$, then $U(\mathcal{Z}'') \subset U(\mathcal{Z}')$. The subspace $U(\mathcal{Z})$ having finite dimension, there exists a subset $\mathcal{Z}_1 \subset \mathcal{Z}$ that is piecewise syndetic, and such that for all piecewise syndetic set $\mathcal{Z}' \subset \mathcal{Z}_1$, one has $U(\mathcal{Z}') = U(\mathcal{Z}_1)$. Let B denote a bound for $\mathcal{Z}_1$ and set

$$\mathcal{E}_0 := \{\boldsymbol{e}(l,m) : m \in [m_0, m_0+B], l \in \mathcal{Z}_1, l+m \in \mathcal{Z}_1\}\,,$$

where m_0 is defined as in the first part of the proof (just before (5.5)). This is a finite set. Let

$$\mathcal{Z}_2 := \{l \in \mathcal{Z}_1 : \exists m \in [m_0, m_0+B] \text{ such that } l+m \in \mathcal{Z}_1\}\,.$$

By Lemma 5.2, the set $\mathcal{Z}_2$ is piecewise syndetic. Now, given $\boldsymbol{e} = \boldsymbol{e}(l,m) \in \mathcal{E}_0$, we set

$$\mathcal{Z}_{\boldsymbol{e}} := \{l \in \mathcal{Z}_2 : T_{\boldsymbol{e}}(T_{\boldsymbol{k}_l}(\boldsymbol{x})) = T_{\boldsymbol{k}_{l+m}}(\boldsymbol{x}) \in U(\mathcal{Z}_1)\}\,.$$

If $l \in \mathcal{Z}_2$, then there exists $m \in [m_0, m_0+b]$ such that $l \in \mathcal{Z}_{\boldsymbol{e}}$ for some $\boldsymbol{e} = \boldsymbol{e}(l,m)$. Hence, $\mathcal{Z}_2 \subset \cup_{\boldsymbol{e}\in\mathcal{E}_0}\mathcal{Z}_{\boldsymbol{e}}$. Since $\mathcal{Z}_2$ is piecewise syndetic, Lemma 5.2 ensures the existence of $\boldsymbol{e} \in \mathcal{E}_0$ such that $\mathcal{Z}_{\boldsymbol{e}}$ is piecewise syndetic. Furthermore, $\mathcal{Z}_{\boldsymbol{e}} \subset \mathcal{Z}_1$. Thus we obtain that

$$U(\mathcal{Z}_{\boldsymbol{e}}) = U(\mathcal{Z}_1)\,.$$

Hence, the vector space $U(\mathcal{Z}_1)$ is invariant under $T_{\boldsymbol{e}}$. Indeed, if $T_{\boldsymbol{k}_l}(\boldsymbol{x}) \in U(\mathcal{Z}_1) = U(\mathcal{Z}_{\boldsymbol{e}})$, then $T_{\boldsymbol{e}}(T_{\boldsymbol{k}_l}(\boldsymbol{x})) \in U(\mathcal{Z}_1)$. By (5.6), there is a decomposition of the form

$$U(\mathcal{Z}_1) = \bigoplus_{i=1}^{r} \iota_i(U_i)\,,$$

where, for every i, $U_i = \pi_i(U(\mathcal{Z}_1)) \subset \mathbb{C}^{n_i}$ is a $T_i^{e_i}$-invariant vector space defined over $\mathbb{Q}$. Since $U(\mathcal{Z}_1)$ is a proper subspace of $\mathbb{C}^N$, there exists i, $1 \leq i \leq r$, such that U_i is a proper subspace of $\mathbb{C}^{n_i}$. This vector space being defined over $\mathbb{Q}$, it has a nonzero vector $\boldsymbol{\nu}_0 \in \mathbb{Z}^{n_i}$ in its orthogonal complement. We thus have

$$\langle \boldsymbol{\nu}_0\,,\, T_i^{e_i k_{i,l}}(\boldsymbol{x}_i)\rangle = 0\,, \tag{5.7}$$

for all $l \in \mathcal{Z}_1$, where $\boldsymbol{x}_i := \pi_i(\boldsymbol{x})$. The set $\mathcal{Z}_1$ being piecewise syndetic, we infer from the definition of the sequence $\boldsymbol{k}_l$ that the set $\mathcal{Z}_2 := \{k_{i,l} : l \in \mathcal{Z}_1\}$ is also piecewise syndetic. By property (iv) of Lemma 5.2, it contains arbitrarily long arithmetic progressions. Let us consider an arithmetic progression of length n_i in $\mathcal{Z}_2$, say

$$a, a+b, a+2b, \ldots, a+(n_i-1)b\,,$$

where $a, b \in \mathbb{N}$. Let us also consider the sequence of vector spaces

$$V_0 \subset \cdots \subset V_{n_i-1} \subset \boldsymbol{\nu}_0^{\perp}$$

defined by

$$V_j = \mathrm{Vect}_{\mathbb{Q}}\left\{T_i^{e_i a}(\boldsymbol{x}_i), \ldots, T_i^{e_i(a+jb)}(\boldsymbol{x}_i)\right\}\,.$$

Since $\dim V_{n_i-1} < n_i$, there exists j_0 such that $V_{j_0} = V_{j_0+1}$. The vector space V_{j_0} is then invariant under $T^{e_i b}$ and we get that $\langle \boldsymbol{\nu}_0\,,\, T^{e_i(a+kb)}(\boldsymbol{x}_i)\rangle = 0$, or equivalently that

$$\left(T_i^{e_i(a+kb)}\boldsymbol{\alpha}_i\right)^{\boldsymbol{\nu}_0} = 1\,, \text{ for all } k \in \mathbb{N}\,.$$

Hence $\boldsymbol{\alpha}_i$ is not T_i-independent. By Theorem 4.6, this provides a contradiction with the assumption that the pair $(T_i, \boldsymbol{\alpha}_i)$ is admissible. □

Lemma 5.7. *We continue with the notation of Theorem 5.4. Let* $\gamma \in \overline{\mathbb{Q}}^\star$ *and* $\boldsymbol{\mu}$ *be a nonzero integer* N*-tuple. Then the set*

$$\mathcal{Z} := \{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}} = \gamma\}$$

is negligible.

Proof. Let us assume by contradiction that $\mathcal{Z}$ is piecewise syndetic and let B be a bound for $\mathcal{Z}$. Set

$$\mathcal{E} := \{\boldsymbol{e}(l,m) : l \in \mathcal{Z}, l+m \in \mathcal{Z}, m \leq B\}\,.$$

This is a finite set. For every $\boldsymbol{e} \in \mathcal{E}$, set

$$\mathcal{Z}_{\boldsymbol{e}} := \left\{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}-\boldsymbol{\mu}T_{\boldsymbol{e}}} = 1\right\}$$

and

$$\mathcal{Z}' := \{l \in \mathcal{Z} : \exists m \leq B \text{ such that } l+m \in \mathcal{Z}\}\,.$$

Lemma 5.2 implies that $\mathcal{Z}'$ is piecewise syndetic. For $l \in \mathcal{Z}'$, there exists $\boldsymbol{e} = e(l,m) \in \mathcal{E}$ such that $m \leq B$ and $l+m \in \mathcal{Z}$. Thus,

$$(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}-\boldsymbol{\mu}T_{\boldsymbol{e}}} = \gamma/\gamma = 1$$

and we get that $\mathcal{Z}' \subset \cup_{\boldsymbol{e}\in\mathcal{E}}\mathcal{Z}_{\boldsymbol{e}}$. Lemma 5.2 ensures the existence of $\boldsymbol{e} \in \mathcal{E}$ such that $\mathcal{Z}_{\boldsymbol{e}}$ is piecewise syndetic. By Lemma 5.6, it follows that $\boldsymbol{\mu} - T_{\boldsymbol{e}}\boldsymbol{\mu} = 0$ for such a vector $\boldsymbol{e}$, which contradicts the assumption that none of the eigenvalues of T_i is a root of unity. □

We are now ready to prove Theorem 5.4.

Proof of Theorem 5.4. Set

$$\mathcal{Z} := \{l \in \mathbb{N} : g(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\}\ .$$

The assumption that the pairs $(T_i, \boldsymbol{\alpha}_i)$ are admissible allows us to apply [22, Theorem 3] to the sequence of points $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$. In order to apply the result of Corvaja and Zannier, we need to prove that the following three conditions are satisfied.

(i) There exists a finite set of places $\mathcal{S}$ such that the algebraic points $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$ are $\mathcal{S}$-units.
(ii) The sequence $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ tends to 0.
(iii) One has $\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(-\log\|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\|)$, where we let H denote the absolute Weil height (see Section 3).

Condition (i) is easy to check. Indeed, any finite number of nonzero algebraic numbers are $\mathcal{S}$-units for some $\mathcal{S}$. The coordinates of the vector $\boldsymbol{\alpha}$ are thus $\mathcal{S}$-units for some fixed $\mathcal{S}$, and it follows directly that all $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$ are $\mathcal{S}$-units too. Since by assumption the pairs $(T_i, \boldsymbol{\alpha}_i)$ are admissible, Theorem 4.6 implies that the sequence $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ tends to 0, and thus (ii) is satisfied. Next we check that (iii) holds. We infer from Lemma 5.5 that

$$\|T_{\boldsymbol{k}_l}\| = \mathcal{O}(e^l) \quad\text{and}\quad \log\|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\| \leq -ce^l\,,$$

for some positive real number c and all sufficiently large integers l. On the other hand, we have $\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(\|T_{\boldsymbol{k}_l}\|)$. It thus follows that

$$\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(-\log\|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\|)\,,$$

which shows that Condition (iii) is satisfied.

Applying [22, Theorem 3] to the sequence of algebraic points $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ and to the function $g(\boldsymbol{z})$, we obtain the existence of a finite number of N-tuples $\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_s$ and of algebraic numbers $\gamma_1,\ldots,\gamma_s$, such that

$$\mathcal{Z}\subset\bigcup_{i=1}^{s}\mathcal{Z}_i$$

where

$$\mathcal{Z}_i:=\{l\in\mathbb{N}:(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}_i}=\gamma_i\}\,.$$

By Lemma 5.7, the sets $\mathcal{Z}_i$ are all negligible. It thus follows from Lemma 5.3 that $\mathcal{Z}$ is also negligible, which proves the theorem. $\square$

6. Mahler's method in families

In this section, we state Theorem 6.2, a general lifting theorem dealing with families of Mahler systems associated with sufficiently independent transformations. In Section 9, Theorems 2.3, 2.6, and 2.8, as well as Corollary 2.9, will be deduced from this result.

6.1. **Statement of Theorem 6.2.** Let r be a positive integer. For every i, $1\leq i\leq r$, let us consider a Mahler system

$$\begin{pmatrix} f_{i,1}(\boldsymbol{z}_i)\\ \vdots\\ f_{i,m_i}(\boldsymbol{z}_i)\end{pmatrix}=A_i(\boldsymbol{z}_i)\begin{pmatrix} f_{i,1}(T_i\boldsymbol{z}_i)\\ \vdots\\ f_{i,m_i}(T_i\boldsymbol{z}_i)\end{pmatrix}\tag{6.1.i}$$

where n_i and m_i are positive integers, $\boldsymbol{z}_i:=(z_{i,1},\ldots,z_{i,n_i})$ is a vector of indeterminates, T_i is an $n_i\times n_i$ matrix with nonnegative integer coefficients and with spectral radius $\rho(T_i)$, $A_i(\boldsymbol{z}_i)$ belongs to $\in\mathrm{GL}_{m_i}(\overline{\mathbb{Q}}(\boldsymbol{z}_i))$, and $f_{i,1}(\boldsymbol{z}_i),\ldots,f_{i,m_i}(\boldsymbol{z}_i)$ belong to $\overline{\mathbb{Q}}\{\boldsymbol{z}_i\}$. We also let $\boldsymbol{\alpha}_i\in(\overline{\mathbb{Q}}^{\star})^{n_i}$ and $\boldsymbol{X}_i:=(X_{i,1},\ldots,X_{i,m_i})$ denote a vector of indeterminates. Set $\boldsymbol{z}:=(\boldsymbol{z}_1,\ldots,\boldsymbol{z}_r)$ and $\boldsymbol{\alpha}:=(\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r)$.

Remark 6.1. Note that one has to replace $A_i(\boldsymbol{z}_i)$ by $A_i(\boldsymbol{z}_i)^{-1}$ to obtain a system as in (2.2). However, it is more natural in our proof to work with systems written in the form (6.1.i). We recall that $\overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$ denote the algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ in $\overline{\mathbb{Q}}\{\boldsymbol{z}-\boldsymbol{\alpha}\}$.

Theorem 6.2. *We continue with the above assumptions. Let us assume that the two following conditions hold.*

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t.* (6.1.i) *and $(T_i,\boldsymbol{\alpha}_i)$ is admissible.*
(ii) *$\rho(T_1),\ldots,\rho(T_r)$ are pairwise multiplicatively independent.*

Then for every polynomial $P\in\overline{\mathbb{Q}}[\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r]$ that is homogeneous with respect to each family of indeterminates $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r$, and such that

$$P(f_{1,1}(\boldsymbol{\alpha}_1),\ldots,f_{r,m_r}(\boldsymbol{\alpha}_r))=0\,,$$

there exists a polynomial $Q\in\overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}[\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r]$, homogeneous with respect to each family of indeterminates $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r$, and such that

$$Q(\boldsymbol{z},f_{1,1}(\boldsymbol{z}_1),\ldots,f_{r,m_r}(\boldsymbol{z}_r))=0\ \text{and}\ Q(\boldsymbol{\alpha},\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r)=P(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r)\,.$$

Furthermore, if $\overline{\mathbb{Q}}(\boldsymbol{z})(f_{1,1}(\boldsymbol{z}_1),\ldots,f_{r,m_r}(\boldsymbol{z}_r))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$, then there exists such a polynomial Q in $\overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r]$.

6.2. **Notation.** In order to lighten the notation, we let $\boldsymbol{f}_i(\boldsymbol{z}_i)$ denote the column vector formed by the functions $f_{i,1}(\boldsymbol{z}_i), \ldots, f_{i,m_i}(\boldsymbol{z}_i)$. We also set

$$M := \sum_{i=1}^{r} m_i \qquad \text{and} \qquad N := \sum_{i=1}^{r} n_i. \tag{6.2}$$

Iterating k times the system (6.1.i), one obtains the new system

$$\boldsymbol{f}_i(\boldsymbol{z}_i) = A_{i,k}(\boldsymbol{z}_i)\boldsymbol{f}_i(T_i^k \boldsymbol{z}_i)\,, \tag{6.3.i}$$

where

$$A_{i,k}(\boldsymbol{z}_i) := A_i(\boldsymbol{z}_i)A_i(T_i\boldsymbol{z}_i)\cdots A_i(T_i^{k-1}\boldsymbol{z}_i)\,.$$

Set $\boldsymbol{z} := (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r)$ and by abuse of notation $\boldsymbol{f}_i(\boldsymbol{z}) := \boldsymbol{f}_i(\boldsymbol{z}_i)$. For every r-tuple of positive integers $\boldsymbol{k} = (k_1, \ldots, k_r)$, one can gather the systems (6.3.i) into a single one as follows:

$$\begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_r(\boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} A_{1,k_1}(\boldsymbol{z}_1) & & \\ & \ddots & \\ & & A_{r,k_r}(\boldsymbol{z}_r) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1(T_{\boldsymbol{k}}\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_r(T_{\boldsymbol{k}}\boldsymbol{z}) \end{pmatrix}, \tag{6.4}$$

where $T_{\boldsymbol{k}} := T_1^{k_1} \oplus \cdots \oplus T_r^{k_r}$. Finally, we let $\boldsymbol{f}(\boldsymbol{z})$ denote the column vector formed by all functions $f_{i,j}(\boldsymbol{z}_i)$, and $A_{\boldsymbol{k}}(\boldsymbol{z})$ denote the block diagonal matrix defined so that (6.4) can be shortened to

$$\boldsymbol{f}(\boldsymbol{z}) = A_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z})\,. \tag{6.5}$$

We keep this notation for the rest of the paper.

6.3. **Choice of the sequence $\boldsymbol{k}_l$.** In order to prove Theorem 6.2, one needs to choose a sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset \mathbb{N}^r$ satisfying the asymptotic

$$\boldsymbol{k}_l = \Theta l + \mathcal{O}(1), \qquad l \to \infty\,, \tag{6.6}$$

where Θ is defined as in (5.1), as well as some additional properties. Let us define a partial order over $\mathbb{Z}^r$ by setting $\boldsymbol{k}_1 \preceq \boldsymbol{k}_2$ when the vector $\boldsymbol{k}_2 - \boldsymbol{k}_1$ has nonnegative coordinates. Let V denote the orthogonal complement to the vector Θ in $\mathbb{Z}^r$. That is,

$$V := \{\boldsymbol{\mu} \in \mathbb{Z}^r \,:\, \langle \boldsymbol{\mu}, \Theta\rangle = 0\}\,.$$

Let $V^\perp$ denote the orthogonal complement to V in $\mathbb{Z}^r$. That is,

$$V^\perp := \{\boldsymbol{k} \in \mathbb{Z}^r \,:\, \langle \boldsymbol{k}, \boldsymbol{\mu}\rangle = 0 \text{ for all } \boldsymbol{\mu} \in V\}\,.$$

We also set $V_+^\perp := V^\perp \cap \mathbb{N}^r$.

Lemma 6.3. *There exists a sequence of nonnegative r-tuples $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset V_+^\perp$, satisfying the asymptotic* (6.6), *and such that $\boldsymbol{k}_l \preceq \boldsymbol{k}_{l+1}$ for all $l \in \mathbb{N}$.*

Proof. We first note that the set V could possibly be reduced to $\{0\}$; this is the case when the numbers

$$1/\log\rho(T_1), \ldots, 1/\log\rho(T_r)$$

are linearly independent over the rational numbers[6]. In that case, one can simply choose

$$\boldsymbol{k}_l := \left(\left\lfloor \frac{l}{\rho(T_1)} \right\rfloor, \ldots, \left\lfloor \frac{l}{\rho(T_r)} \right\rfloor \right) .$$

In contrast, the set $V^\perp$ is a nonempty $\mathbb{Z}$-module. Indeed, the $\mathbb{R}$-vector space generated by $V^\perp$ in $\mathbb{R}^r$ contains the vector Θ. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s$ be a $\mathbb{Z}$-basis of $V^\perp$. For all $l \in \mathbb{N}$, there exist real numbers $\lambda_1(l), \ldots, \lambda_s(l)$ such that

$$l\Theta = \lambda_1(l)\boldsymbol{e}_1 + \cdots + \lambda_s(l)\boldsymbol{e}_s .$$

We deduce that

$$\|l\Theta - \lfloor\lambda_1(l)\rfloor\boldsymbol{e}_1 + \cdots + \lfloor\lambda_s(l)\rfloor\boldsymbol{e}_s\| \leq \sum_{i=1}^{s} \|\boldsymbol{e}_i\| . \tag{6.7}$$

Since all coordinates of Θ are positive, there exists a nonnegative integer l_0 such that, for all $l \geq l_0$, the vector $\lfloor\lambda_1(l)\rfloor\boldsymbol{e}_1 + \cdots + \lfloor\lambda_s(l)\rfloor\boldsymbol{e}_s$ has positive coordinates. For every $l \geq l_0$, set

$$\boldsymbol{a}_l := \lfloor\lambda_1(l)\rfloor\boldsymbol{e}_1 + \cdots + \lfloor\lambda_s(l)\rfloor\boldsymbol{e}_s \in V_+^\perp .$$

The sequence $(\boldsymbol{a}_l)_{l \geq l_0}$ agrees with the asymptotic (6.6), but not necessarily with the order $\preceq$. Let $e := \sum_{i=1}^{s} \|\boldsymbol{e}_i\|$ and let $\theta > 0$ denote the minimum of the modules of the coordinates of the vector Θ. If $l_1 \geq l_0$ and $l_2 \geq l_1 + 2e/\theta$, then (6.7) implies that $\boldsymbol{a}_{l_1} \preceq \boldsymbol{a}_{l_2}$. Set $b := \lceil 2e/\theta \rceil$. Let us define the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset \mathbb{N}^r$ by setting

$$\boldsymbol{k}_{l_0+lb+j} := \boldsymbol{a}_{l_0+lb}$$

for $l \in \mathbb{N}$ and $0 \leq j < b$, and $\boldsymbol{k}_l = \boldsymbol{0}$ for $l < l_0$. Then the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ has all the required properties. ☐

7. Hilbert's Nullstellensatz and relation matrices

In this section, we gather some preliminary results needed for proving Theorem 6.2. In particular, we introduce the so-called *relation matrices* and study some of their properties.

Let us fix some notation. Let r, and $m_1, \ldots, m_r$, be some positive integers, and let $d_1, \ldots, d_r$ be some nonnegative integers. Set $M = m_1 + \cdots + m_r$ and let $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_t$ denote an enumeration of all distinct M-tuples $\boldsymbol{\mu} := (\mu_{1,1}, \ldots, \mu_{1,m_1}, \mu_{2,1}, \ldots, \mu_{r,m_r})$ such that

$$\mu_{i,1} + \cdots + \mu_{i,m_i} = d_i ,$$

for every i, $1 \leq i \leq r$. For every i, $1 \leq i \leq r$, we let B_i denote an $m_i \times m_i$ matrix with coefficients in some commutative ring $\mathcal{R}$, and we let $B = B_1 \oplus \cdots \oplus B_r$. Given a vector of indeterminates $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$, where $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,m_i})$, we note that $(B(\boldsymbol{X}))^{\boldsymbol{\mu}_j} \in \mathcal{R}[\boldsymbol{X}]$ is a homogeneous polynomial of degree d_i in each set of variables $\boldsymbol{X}_i$. We let $R_{j,l}(B)$ denote the elements of $\mathcal{R}$ defined by

$$(B(\boldsymbol{X}))^{\boldsymbol{\mu}_j} = \sum_{l=1}^{t} R_{j,l}(B)\boldsymbol{X}^{\boldsymbol{\mu}_l} . \tag{7.1}$$

[6]When $r > 2$, It is not known whether the pairwise multiplicative independence of the numbers $\rho(T_i)$ implies that their reciprocals are linearly independent over $\mathbb{Q}$.

For every j and l, $R_{j,l}(B)$ is a polynomial of degree $d := \max\{d_1, \ldots, d_r\}$ in the coefficients of the matrix B. Set $R(B) := (R_{j,l}(B))_{1\leq j,l\leq t}$. Let B_1 and B_2 be two $M \times M$ block diagonal matrix. Then

$$R(B_1B_2) = R(B_1)R(B_2) \tag{7.2}$$

and

$$R(B_1)^{-1} = R(B_1^{-1}) \tag{7.3}$$

when B_1 is invertible.

All along this section, we continue with the notation of Section 6.2. We let $A_{\boldsymbol{k}}(\boldsymbol{z}) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$ and $f_{1,1}(\boldsymbol{z}), \ldots, f_{r,m_r}(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be as in (6.5). We fix some $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r) \in (\overline{\mathbb{Q}}^\star)^N$ such that $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^\star)^{n_i}$ is regular with respect to the matrix A_i. We also fix a sequence of integer vectors $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ satisfying the assumptions of Lemma 6.3. Then, for every $\boldsymbol{k} \in \mathbb{N}^r$, we set

$$\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) := R(A_{\boldsymbol{k}}(\boldsymbol{z}))\,. \tag{7.4}$$

This is a $t \times t$-matrix with coefficients in $\overline{\mathbb{Q}}(\boldsymbol{z})$. One has $\boldsymbol{R}_{\boldsymbol{0}}(\boldsymbol{z}) = \mathrm{I}_t$ and, given $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{N}^r$, it follows from (7.2) that

$$\boldsymbol{R}_{\boldsymbol{k}_1+\boldsymbol{k}_2}(\boldsymbol{z}) = \boldsymbol{R}_{\boldsymbol{k}_1}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_2}(T_{\boldsymbol{k}_1}\boldsymbol{z})\,. \tag{7.5}$$

Furthermore, since the points $\boldsymbol{\alpha}_i$ are assumed to be regular, we infer from (7.3) that the matrix $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{\alpha})$ is well-defined and invertible for all $\boldsymbol{k} \in \mathbb{N}^r$, with inverse equal to $R(A_{\boldsymbol{k}}(\boldsymbol{\alpha})^{-1})$.

Let $\boldsymbol{Y} := (y_{i,j})_{1\leq i,j\leq t}$ denote a matrix of indeterminates. Given a field $\mathbb{K}$ and a nonnegative integer δ_1, we let $\mathbb{K}[\boldsymbol{Y}]_{\delta_1}$ denote the set of polynomials of degree at most δ_1 in every indeterminate $y_{i,j}$. Given two nonnegative integers δ_1 and δ_2, we let $\mathbb{K}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$ denote the set of polynomials $P \in \mathbb{K}[\boldsymbol{Y}, \boldsymbol{z}]$ of degree at most δ_1 in every indeterminate $y_{i,j}$ and of total degree at most δ_2 in the indeterminates $z_{i,j}$. By Theorem 5.4, a polynomial $P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ is well-defined at the point $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all l in a full subset of $\mathbb{N}$. Set

$$\mathcal{I} := \{P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}] \ : P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,,\ \forall l \text{ in a full subset of } \mathbb{N}\}\,.$$

Piecewise syndetic, negligible, and full sets are introduced in Definition 5.1.

7.1. **Estimates for the dimension of certain vector spaces.** Let δ_1 and δ_2 be two nonnegative integers. Set $\mathcal{I}(\delta_1) := \mathcal{I} \cap \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]_{\delta_1}$ and $\mathcal{I}(\delta_1, \delta_2) := \mathcal{I} \cap \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$. Note that $\mathcal{I}(\delta_1, \delta_2)$ is a vector subspace of $\overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$, and let $\mathcal{I}^\perp(\delta_1, \delta_2)$ denote a complement to $\mathcal{I}(\delta_1, \delta_2)$ in $\overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$.

Lemma 7.1. *Let $d(\delta_1, \delta_2)$ denote the dimension of $\mathcal{I}^\perp(\delta_1, \delta_2)$ over $\overline{\mathbb{Q}}$. There exists a positive real number $c_1(\delta_1)$, that does not depend on δ_2, such that*

$$d(\delta_1, \delta_2) \sim c_1(\delta_1)\delta_2^N\,,\ \text{as } \delta_2 \text{ tends to infinity.}$$

Proof. Let $\boldsymbol{Y}^{\boldsymbol{\nu}_1}, \ldots, \boldsymbol{Y}^{\boldsymbol{\nu}_h}$, $h := (\delta_1 + 1)^{t^2}$, denote an enumeration of the monomials of degree at most δ_1 in every indeterminates $y_{i,j}$. Let $P_1, \ldots, P_h$ be polynomials in $\mathcal{I}(\delta_1)$. Every P_i has a unique decomposition of the form

$$P_i(\boldsymbol{Y}, \boldsymbol{z}) := \sum_{j=1}^{h} p_{i,j}(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j}\,,$$

where $p_{i,j}(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$, $1 \leq i,j \leq h$. Let $C(\boldsymbol{z})$ denote the square matrix defined by $C(\boldsymbol{z}) := (p_{i,j}(\boldsymbol{z}))_{1\leq i,j\leq h}$. By Theorem 5.4, $C(\boldsymbol{z})$ is well-defined at $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$ for all l in a full set $\mathcal{Z}_0 \subset \mathbb{N}$. For every i, $1 \leq i \leq h$, let $\mathcal{Z}_i$ denote the set of nonnegative integers l such that $P_i(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$. Since $P_1, \ldots, P_h \in \mathcal{I}(\delta_1)$, the sets $\mathcal{Z}_1, \ldots, \mathcal{Z}_h$ are full. By Lemma 5.3, the set $\mathcal{Z} := \bigcap_{i=0}^h \mathcal{Z}_i$ is full. For every $l \in \mathcal{Z}$, one has

$$C(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_1} \\ \vdots \\ \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_h} \end{pmatrix} = \begin{pmatrix} P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ \vdots \\ P_h(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \end{pmatrix} = 0\,. \tag{7.6}$$

Since the matrix $\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$ is invertible, it is nonzero. Hence the vector $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_1}, \ldots, \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_h})$ is also nonzero, and we deduce from (7.6) that $\det C(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ for all $l \in \mathcal{Z}$. Since $\mathcal{Z}$ is not negligible, Theorem 5.4 implies that $\det C(\boldsymbol{z}) = 0$. Hence $\mathcal{I}(\delta_1)$ is a strict subspace of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]_{\delta_1}$, say of dimension $d < h$. There thus exist polynomials $b_{i,j}(z) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$, $1 \leq i \leq h-d$, $1 \leq j \leq h$, such that for all $p_1(\boldsymbol{z}), \ldots, p_h(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$:

$$\sum_{j=1}^h p_j(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \mathcal{I}(\delta_1) \Leftrightarrow \sum_{j=1}^h b_{i,j}(\boldsymbol{z})p_j(\boldsymbol{z}) = 0 \quad \forall i,\ 1 \leq i \leq h-d\,. \tag{7.7}$$

Since $d + (h-d) = h$, these equations are linearly independent. Now, let us consider a polynomial $P := \sum_{j=1}^h p_j(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$ and set

$$p_j(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\,:\,|\boldsymbol{\lambda}|\leq\delta_2} p_{j,\boldsymbol{\lambda}}\boldsymbol{z}^{\boldsymbol{\lambda}} \quad\text{and}\quad b_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{\kappa}\,:\,|\boldsymbol{\kappa}|\leq\delta_1'} b_{i,j,\boldsymbol{\kappa}}\boldsymbol{z}^{\boldsymbol{\kappa}}\,,$$

where δ_1' is a nonnegative integer depending only on δ_1, and where the numbers $p_{j,\boldsymbol{\lambda}}$ and $b_{i,j,\boldsymbol{\kappa}}$ are algebraic for all quadruples $(i, j, \boldsymbol{\lambda}, \boldsymbol{\kappa})$. By (7.7), P belongs to $\mathcal{I}(\delta_1, \delta_2)$ if and only if

$$\sum_{j=1}^h \sum_{\substack{|\boldsymbol{\lambda}|\leq\delta_2, |\boldsymbol{\kappa}|\leq\delta_1' \\ \boldsymbol{\lambda}+\boldsymbol{\kappa}=\boldsymbol{\gamma}}} b_{i,j,\boldsymbol{\kappa}}p_{j,\boldsymbol{\lambda}} = 0\,, \quad \forall(\boldsymbol{\gamma}, i)\,.$$

If we fix the vector $\boldsymbol{\gamma}$, then, as soon as every coordinate of $\boldsymbol{\gamma}$ is larger than δ_1' and that $|\boldsymbol{\gamma}| \leq \delta_2$, the number of linearly independent equations does not depend on our choice for $\boldsymbol{\gamma}$. Thus, the total number of linearly independent equations defining the vector space $\mathcal{I}(\delta_1, \delta_2)$ is equivalent to

$$c_1(\delta_1)\delta_2^N\,,$$

when δ_2 tends to infinity, where $c_1(\delta_1)$ is a positive real number depending only on δ_1. This number is precisely the dimension of the vector space $\mathcal{I}^\perp(\delta_1, \delta_2)$, which ends the proof. □

Lemma 7.2. *For every pair of nonnegative integers (δ_1, δ_2), one has*

$$\dim \mathcal{I}^\perp(2\delta_1, \delta_2) \leq 2^{t^2} \dim \mathcal{I}^\perp(\delta_1, \delta_2)\,.$$

Proof. Let $P(\boldsymbol{Y},\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,\delta_2}$. Such a polynomial has a decomposition of the form

$$P(\boldsymbol{Y},\boldsymbol{z}) = \sum_{\ell=1}^{t^2} e_\ell(\boldsymbol{Y})^{\delta_1} P_\ell(\boldsymbol{Y},\boldsymbol{z}) \,, \tag{7.8}$$

where we let $e_\ell(\boldsymbol{Y})$ denote the 2^{t^2} monomials of degree at most one in the indeterminates $y_{i,j}$, and where each $P_\ell(\boldsymbol{Y},\boldsymbol{z})$ belongs to $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$. If, in such a decomposition, every polynomial P_ℓ belongs to $\mathcal{I}(\delta_1,\delta_2)$, then P belongs to $\mathcal{I}(2\delta_1,\delta_2)$. The decomposition (7.8) naturally defines a surjective linear map from $\left(\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}/\mathcal{I}(\delta_1,\delta_2)\right)^{2^{t^2}}$ to $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,\delta_2}/\mathcal{I}(2\delta_1,\delta_2)$. It follows that

$$\dim_{\overline{\mathbb{Q}}} \mathcal{I}^\perp(2\delta_1,\delta_2) \leq 2^{t^2} \dim_{\overline{\mathbb{Q}}} \mathcal{I}^\perp(\delta_1,\delta_2) \,,$$

as wanted. $\square$

7.2. **Nullstellensatz and relation matrices.** In this section, we show how Hilbert's Nullstellensatz allows us to exhibit a matrix $\boldsymbol{\phi}$, called a *relation matrix*, whose coordinates are all algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$, and which encodes the algebraic relations over $\overline{\mathbb{Q}}(\boldsymbol{z})$ of degree at most d_i in each variables between the functions $f_{1,1}(\boldsymbol{z}),\ldots,f_{r,m_r}(\boldsymbol{z})$. These relation matrices are the cornerstone of the proof of Theorem 6.2.

We first prove the following lemma.

Lemma 7.3. *The set $\mathcal{I}$ is a radical ideal of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$.*

Proof. Checking that $\mathcal{I}$ is an ideal of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ is not difficult. If $P_1, P_2 \in \mathcal{I}$, and if $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) are full sets of nonnegative integers l for which $P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ (resp. $P_2(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$), then $P_1 + P_2$ vanishes at the points $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all $l \in \mathcal{Z}_1 \cap \mathcal{Z}_2$. By Lemma 5.3, this set is full. Hence $P_1 + P_2 \in \mathcal{I}$.

Now let $P_1 \in \mathcal{I}$ and $P_2 \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$. On the one hand, $P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ for all l in a full set $\mathcal{Z}_1$, while, on the other hand, Theorem 5.4 ensures that $P_2(\boldsymbol{Y},\boldsymbol{z})$ is well-defined at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all nonnegative integers l outside a negligible set $\mathcal{Z}_2$. We deduce that

$$P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) P_2(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$$

for all $l \in \mathcal{Z}_1 \setminus \mathcal{Z}_2$. By Lemma 5.3, this is a full set. Hence $P_1P_2 \in \mathcal{I}$.

Let $P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ be such that $P^r \in \mathcal{I}$ for some r. If l is a nonnegative integer such that $P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^r = 0$, then $P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$. Hence $P \in \mathcal{I}$ and $\mathcal{I}$ is a radical ideal. $\square$

Let $\mathbb{K}$ denote an algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$.

Lemma 7.4. *There exists a matrix $\boldsymbol{\phi} \in \mathrm{GL}_t(\mathbb{K})$ such that*

$$P(\boldsymbol{\phi},\boldsymbol{z}) = 0 \,,$$

for all polynomials $P \in \mathcal{I}$.

Proof. Let us consider the affine algebraic set $\mathcal{V}$ associated with the radical ideal $\mathcal{I}$. That is,

$$\mathcal{V} := \{\boldsymbol{\phi} \in \mathcal{M}_t(\mathbb{K}) \ : \ P(\boldsymbol{\phi},\boldsymbol{z}) = 0 \,,\ \forall P \in \mathcal{I}\} \,.$$

According to the weak form of Hilbert's Nullstellensatz (see, for instance, [33, Theorem 1.4, p. 379]), $\mathcal{V}$ is nonempty as soon as $\mathcal{I}$ is a proper ideal of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$. But the definition of $\mathcal{I}$ clearly implies that nonzero constant polynomials do not belong to $\mathcal{I}$. Hence $\mathcal{V}$ is nonempty.

Now, let us assume by contradiction that $\det \boldsymbol{\phi} = 0$ for all $\boldsymbol{\phi}$ in $\mathcal{V}$. By Hilbert's Nullstellensatz (see, for instance, [33, Theorem 1.5, p. 380]), the polynomial $\det \boldsymbol{Y}$ belongs to the radical of the ideal $\mathcal{I}$. Hence $\det \boldsymbol{Y} \in \mathcal{I}$ for $\mathcal{I}$ is radical. Thus, $\det \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}) = 0$ for all l in a full set. This provides a contradiction since $\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$ is invertible for all l in $\mathbb{N}$. We thus deduce that there exists an invertible matrix $\boldsymbol{\phi}$ in $\mathcal{V}$, as wanted. □

Definition 7.5. A matrix $\boldsymbol{\phi} \in \mathrm{GL}_t(\mathbb{K})$ satisfying the property of Lemma 7.4 is called a *relation matrix*.

The next lemma plays a central role in the proof of Theorem 6.2.

Lemma 7.6. *Let $\boldsymbol{\phi} \in \mathrm{GL}_t(\mathbb{K})$ be a relation matrix. Then*

$$P\left(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}\right) = 0\,,$$

for all $P \in \mathcal{I}$ and all $\boldsymbol{k} \in V_+^{\perp}$.

Proof. Let $\mathcal{A}$ denote the subring of $\overline{\mathbb{Q}}(\boldsymbol{z})$ formed by all rational functions with no pole at the points $T_{\boldsymbol{k}}\boldsymbol{\alpha}$, $\boldsymbol{k} \in \mathbb{N}^r$. The set $\mathcal{S}$ of polynomials $P \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ that does not vanish at any of the points $T_{\boldsymbol{k}}\boldsymbol{\alpha}$, $\boldsymbol{k} \in \mathbb{N}^r$, being multiplicatively closed, the ring $\mathcal{A}$ is the localization of $\overline{\mathbb{Q}}[\boldsymbol{z}]$ at $\mathcal{S}$, *i.e.*, $\mathcal{A} = \mathcal{S}^{-1}\overline{\mathbb{Q}}[\boldsymbol{z}]$. It follows that $\mathcal{A}$ is a Noetherian ring (see, for instance, [33, Proposition 1.6, p. 415]). Since by assumption each of the points $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$ is regular with respect to the corresponding Mahler system (6.1.i), the coefficients of the matrices $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$, $\boldsymbol{k} \in \mathbb{N}^r$, belong to the ring $\mathcal{A}$. With any piecewise syndetic set $\mathcal{Z} \subset \mathbb{N}$, we associate the set

$$\mathcal{I}_{\mathcal{Z}} := \left\{P \in \mathcal{A}[\boldsymbol{Y}] \;:\; P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,,\ \forall l \in \mathcal{Z}\right\}\,.$$

The proof is divided into the following seven simple results, namely Facts 1 to 7.

Fact 1. The set $\mathcal{I}_{\mathcal{Z}}$ is an ideal of $\mathcal{A}[\boldsymbol{Y}]$.

Let $P_1, P_2 \in \mathcal{I}_{\mathcal{Z}}$. Then $P_1 + P_2$ vanishes at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all $l \in \mathcal{Z}$. Hence $P_1 + P_2 \in \mathcal{I}_{\mathcal{Z}}$. Let $P_1 \in \mathcal{I}_{\mathcal{Z}}$ and $P_2 \in \mathcal{A}[\boldsymbol{Y}]$. Then $P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ for all $l \in \mathcal{Z}$. On the other hand, by definition of $\mathcal{A}$, $P_2(\boldsymbol{Y}, \boldsymbol{z})$ has no pole at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$, $l \in \mathbb{N}$. It follows that P_1P_2 vanishes at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all $l \in \mathcal{Z}$. Hence $P_1P_2 \in \mathcal{I}_{\mathcal{Z}}$, which proves Fact 1.

If $\mathcal{Z}'$ is a piecewise syndetic set such that $\mathcal{Z}' \subset \mathcal{Z} \subset \mathbb{N}$, one has $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{I}_{\mathcal{Z}'}$. Since $\mathcal{A}$ is Noetherian, $\mathcal{A}[\boldsymbol{Y}]$ is Noetherian too and any increasing sequence of ideals is stationary. Thus, for every piecewise syndetic set $\mathcal{Z} \subset \mathbb{N}$, there exists a piecewise syndetic set $\mathcal{Z}_0 \subset \mathcal{Z}$ such that $\mathcal{I}_{\mathcal{Z}_1} = \mathcal{I}_{\mathcal{Z}_0}$ for all piecewise syndetic sets $\mathcal{Z}_1 \subset \mathcal{Z}_0$. For Facts 2 to 7, we fix such a pair of sets $(\mathcal{Z}, \mathcal{Z}_0)$.

Fact 2. There exist infinitely many r-tuples $\boldsymbol{k} \in V_+^{\perp}$ such that

$$\mathcal{Z}_0(\boldsymbol{k}) := \{l \in \mathcal{Z}_0 \;:\; \exists l' \in \mathcal{Z}_0,\ \boldsymbol{k}_{l'} = \boldsymbol{k}_l + \boldsymbol{k}\} \tag{7.9}$$

is a piecewise syndetic set.

Let B be a bound for $\mathcal{Z}_0$ and $e \geq 0$ be an integer. By Lemma 5.2, the set

$$\mathcal{Z}_e := \{l \in \mathcal{Z}_0 \ : \ \exists l' \in \mathcal{Z}_0, e \leq l' - l \leq e + B\}$$

is piecewise syndetic. By (6.6), there are only finitely many differences $\boldsymbol{k}_{l'} - \boldsymbol{k}_l$ for which $e \leq l' - l \leq e + B$. Furthermore, by Lemma 6.3, such differences all belong to $V_+^{\perp}$. Let $\mathcal{K}_e \subset V_+^{\perp}$ denote the finite set formed by these differences. Then

$$\mathcal{Z}_e \subset \cup_{\boldsymbol{k} \in \mathcal{K}_e} \mathcal{Z}_0(\boldsymbol{k}) \,.$$

By Lemma 5.2, at least one of the sets $\mathcal{Z}_0(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{K}_e$, is piecewise syndetic. Letting e tend to infinity, this proves Fact 2.

Now, let

$$\mathcal{K} := \{\boldsymbol{k} \in V_+^{\perp} : \ \mathcal{Z}_0(\boldsymbol{k}) \ \text{ is piecewise syndetic}\}$$

denote this infinite set.

Fact 3. The $\mathbb{Z}$-module generated by $\mathcal{K}$ in $\mathbb{Z}^r$ is equal to $V^{\perp}$.

Let W denote the $\mathbb{Z}$-module generated by $\mathcal{K}$ in $\mathbb{Z}^r$. It is enough to show that $W^{\perp}$, its orthogonal complement in $\mathbb{Z}^r$, is equal to V. Since $W \subset V^{\perp}$, we have $V \subset W^{\perp}$. Let us prove the reverse inclusion. Let $\boldsymbol{\lambda} \in W^{\perp}$. Then $\boldsymbol{\lambda}$ is orthogonal to all $\boldsymbol{k} \in \mathcal{K}$. By construction, $\mathcal{K}$ remains at bounded distance from $\mathbb{R}\Theta$. Renormalizing and taking the limit for large vectors $\boldsymbol{k} \in \mathcal{K}$, we get that $\boldsymbol{\lambda}$ is orthogonal to Θ. Hence, $\boldsymbol{\lambda} \in V$. This proves Fact 3.

Given $\boldsymbol{k} \in V_+^{\perp}$, we define an action from the monoid $V_+^{\perp}$ to $\mathcal{A}[\boldsymbol{Y}]$ by:

$$\sigma_{\boldsymbol{k}} : \left\{ \begin{array}{ccc} \mathcal{A}[\boldsymbol{Y}] & \rightarrow & \mathcal{A}[\boldsymbol{Y}] \\ P(\boldsymbol{Y}, \boldsymbol{z}) & \mapsto & P(\boldsymbol{Y}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) \,. \end{array} \right.$$

Note that the map $\sigma_{\boldsymbol{k}}$ is well-defined. Indeed, we already observed that the coordinates of $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ belong to the ring $\mathcal{A}$ for all $\boldsymbol{k} \in \mathbb{N}^r$.

Fact 4. For all $\boldsymbol{k} \in \mathcal{K}$, $\sigma_{\boldsymbol{k}}(\mathcal{I}_{\mathcal{Z}_0}) \subset \mathcal{I}_{\mathcal{Z}_0}$.

Let $P \in \mathcal{I}_{\mathcal{Z}_0}$, $\boldsymbol{k} \in \mathcal{K}$, and $l \in \mathcal{Z}_0(\boldsymbol{k})$. Let $l' \in \mathcal{Z}_0$ be such that $\boldsymbol{k}_{l'} = \boldsymbol{k} + \boldsymbol{k}_l$. Then, we have

$$\begin{aligned} \sigma_{\boldsymbol{k}}(P)(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) &= P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}), T_{\boldsymbol{k}}T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ &= P(\boldsymbol{R}_{\boldsymbol{k}_l+\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}+\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ &= P(\boldsymbol{R}_{\boldsymbol{k}_{l'}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_{l'}}\boldsymbol{\alpha}) \\ &= 0 \,. \end{aligned}$$

Thus, $\sigma_{\boldsymbol{k}}(P)$ belongs to the ideal $\mathcal{I}_{\mathcal{Z}_0(\boldsymbol{k})}$, which is equal to $\mathcal{I}_{\mathcal{Z}_0}$ by minimality. This proves Fact 4.

Fact 5. Let $P \in \mathcal{I}_{\mathcal{Z}_0}$ be such that $P = \sigma_{\boldsymbol{k}}(Q)$ for some $Q \in \mathcal{A}[\boldsymbol{Y}]$ and $\boldsymbol{k} \in \mathcal{K}$. Then $Q \in \mathcal{I}_{\mathcal{Z}_0}$.

Since $\mathcal{Z}_0(\boldsymbol{k})$ is piecewise syndetic, we infer from (6.6) that the set

$$\mathcal{Z}_0(\boldsymbol{k})^{-1} := \{l' \in \mathcal{Z}_0 \ : \ \exists l \in \mathcal{Z}_0, \, \boldsymbol{k}_{l'} = \boldsymbol{k}_l + \boldsymbol{k}\}$$

is also piecewise syndetic. By minimality, we obtain $\mathcal{I}_{\mathcal{Z}_0(\boldsymbol{k})^{-1}} = \mathcal{I}_{\mathcal{Z}_0}$. Let $l' \in \mathcal{Z}_0(\boldsymbol{k})^{-1}$ and let $l \in \mathcal{Z}_0$ be such that $\boldsymbol{k}_{l'} := \boldsymbol{k}_l + \boldsymbol{k}$. Then

$$\begin{aligned} Q(\boldsymbol{R}_{\boldsymbol{k}_{l'}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_{l'}}\boldsymbol{\alpha}) &= Q(\boldsymbol{R}_{\boldsymbol{k}_l+\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}+\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ &= Q(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}), T_{\boldsymbol{k}}T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ &= \sigma_{\boldsymbol{k}}(Q)(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) \\ &= P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) \\ &= 0\,. \end{aligned}$$

Thus, $Q \in \mathcal{I}_{\mathcal{Z}_0(\boldsymbol{k})^{-1}} = \mathcal{I}_{\mathcal{Z}_0}$. This proves Fact 5.

Fact 6. For all $\boldsymbol{k} \in V_+^{\perp}$, $\sigma_{\boldsymbol{k}}(\mathcal{I}_{\mathcal{Z}_0}) \subset \mathcal{I}_{\mathcal{Z}_0}$.

Let $\boldsymbol{k} \in V_+^{\perp}$ and $P \in \mathcal{I}_{\mathcal{Z}_0}$. By Fact 3, there is a decomposition of the form

$$\boldsymbol{k} := \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_u - \boldsymbol{a}_{u+1} - \cdots - \boldsymbol{a}_v$$

with $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_v \in \mathcal{K}$. Using recursively Fact 4 with $\sigma_{\boldsymbol{a}_1}, \ldots, \sigma_{\boldsymbol{a}_u}$, we deduce that $\sigma_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_u}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. On the other hand, we have $\sigma_{\boldsymbol{a}_{u+1}+\cdots+\boldsymbol{a}_v}(\sigma_{\boldsymbol{k}}(P)) = \sigma_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_u}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. Using recursively Fact 5 with $\sigma_{\boldsymbol{a}_{u+1}}, \ldots, \sigma_{\boldsymbol{a}_v}$, we obtain that $\sigma_{\boldsymbol{k}}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. This proves Fact 6.

Fact 7. One has $\mathcal{I}_{\mathcal{Z}_0} \subset \mathcal{I}$.

Let $l_0 \in \mathbb{N}$ denote the smallest element in $\mathcal{Z}_0$. Let $P \in \mathcal{I}_{\mathcal{Z}_0}$ and let $l \geq l_0$ be an integer. By Lemma 6.3, $\boldsymbol{k}_l - \boldsymbol{k}_{l_0} \in V_+^{\perp}$. Set $Q := \sigma_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(P)$. By Fact 6, we obtain that $Q \in \mathcal{I}_{\mathcal{Z}_0}$. Since $l_0 \in \mathcal{Z}_0$, we have

$$P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = Q(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) = 0\,.$$

Hence P vanishes at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$, for all $l \geq l_0$. This proves Fact 7.

We are now ready to conclude the proof of Lemma 7.6. Let $P \in \mathcal{I}$, $\boldsymbol{\phi} \in \mathrm{GL}_t(\mathbb{K})$ be a relation matrix, and $\boldsymbol{k} \in V_+^{\perp}$. Let $b(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ be a nonzero polynomial such that $b(\boldsymbol{z})P(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{Y}]$. In particular, $b(\boldsymbol{z})P(\boldsymbol{Y}, \boldsymbol{z}) \in \mathcal{I} \cap \mathcal{A}[\boldsymbol{Y}]$. Let $\mathcal{Z}$ be the set of integers $l \geq 0$ for which $b(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})P(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$. Since $bP \in \mathcal{I}$, $\mathcal{Z}$ is full and hence piecewise syndetic. There thus exists a piecewise syndetic set $\mathcal{Z}_0 \subset \mathcal{Z}$ satisfying Facts 2 to 7. By definition, $b(\boldsymbol{z})P(\boldsymbol{Y}, \boldsymbol{z}) \in \mathcal{I}_{\mathcal{Z}_0}$. By Fact 6, we also have $\sigma_{\boldsymbol{k}}(bP) \in \mathcal{I}_{\mathcal{Z}_0}$, for all $\boldsymbol{k} \in V_+^{\perp}$. By Fact 7, we deduce that $\sigma_{\boldsymbol{k}}(bP) \in \mathcal{I}$. Then, we infer from Lemma 7.4 that

$$b(T_{\boldsymbol{k}}\boldsymbol{z})P(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) = \sigma_{\boldsymbol{k}}(bP)(\boldsymbol{\phi}, \boldsymbol{z}) = 0\,.$$

Since $T_{\boldsymbol{k}}$ is nonsingular and $b(\boldsymbol{z}) \neq 0$, we obtain $P(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) = 0$. □

7.3. **Analyticity of relation matrices.** Let $\boldsymbol{\phi}$ be a relation matrix. All coordinates of $\boldsymbol{\phi}$ being algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$, they generate a finite extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let $\mathbf{k} \subset \mathbb{K}$ denote this extension and let $\gamma \geq 1$ be the degree of $\mathbf{k}$. Choosing a primitive element ϕ in $\mathbf{k}$, we obtain a decomposition of the form

$$\boldsymbol{\phi} = \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(\boldsymbol{z})\phi^j\,, \tag{7.10}$$

where the matrices $\boldsymbol{\phi}_j(\boldsymbol{z})$, $0 \leq j \leq \gamma - 1$, have coefficients in $\overline{\mathbb{Q}}(\boldsymbol{z})$. The field $\mathbb{K}$ is *a priori* an abstract algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$, but we can easily reduce the situation to the case where the coordinates of $\boldsymbol{\phi}$ are analytic at some suitable point $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$.

Lemma 7.7. *We continue with the previous notation. There exist an integer $l_0 \geq 0$, a neighborhood $\mathcal{V}$ of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, and a function $\varphi(\boldsymbol{z})$ that is analytic on $\mathcal{V}$ and algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$ such that the following properties holds.*

(a) $\|T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}\| < 1$.
(b) *$T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$ belongs to the disc of convergence of $f_{1,1}(\boldsymbol{z}), \ldots, f_{r,m_r}(\boldsymbol{z})$.*
(c) *The matrix*

$$\boldsymbol{\phi}(\boldsymbol{z}) := \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(\boldsymbol{z})\varphi(\boldsymbol{z})^j \in \mathrm{GL}_t(\mathcal{M}er(\mathcal{V}))$$

is a relation matrix. That is, it satisfies Lemmas 7.4 and 7.6.
(d) *For every j, $1 \leq j \leq \gamma - 1$, the coordinates of the matrix $\boldsymbol{\phi}_j(\boldsymbol{z})$ are analytic on $\mathcal{V}$ and the matrix $\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})$ is invertible.*

Proof. By Theorem 4.6, $\lim_{l\to\infty} T_{\boldsymbol{k}_l}\boldsymbol{\alpha} = 0$. This ensures that (a) and (b) are satisfied for all sufficiently large integers l.

Let $P(\boldsymbol{z}, y) \in \overline{\mathbb{Q}}[\boldsymbol{z}, y]$ denote the minimal polynomial of ϕ and let $D(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ denote the discriminant of P, seen as a polynomial in the variable y. Since $\boldsymbol{\phi}$ is a relation matrix, $\det \boldsymbol{\phi}$ is nonzero and algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$. There thus exist polynomials $q_0(\boldsymbol{z}), \ldots, q_\nu(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$, $q_0(\boldsymbol{z}) \neq 0$, such that

$$q_0(\boldsymbol{z}) = q_1(\boldsymbol{z}) \det \boldsymbol{\phi} + q_2(\boldsymbol{z}) \det \boldsymbol{\phi}^2 + \cdots + q_\nu(\boldsymbol{z}) \det \boldsymbol{\phi}^\nu \,. \tag{7.11}$$

Let $d(\boldsymbol{z})$ be the least common multiple of the denominators of the coefficients of the matrices $\boldsymbol{\phi}_0(\boldsymbol{z}), \ldots, \boldsymbol{\phi}_{\gamma-1}(\boldsymbol{z})$. Since the polynomial $D(\boldsymbol{z})q_0(\boldsymbol{z})d(\boldsymbol{z})$ is nonzero, Theorem 5.4 ensures the existence of a full set $\mathcal{E} \subset \mathbb{N}$ such that

$$D(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})q_0(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})d(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0 \,, \ \forall l \in \mathcal{E} \,.$$

Let $l_0 \in \mathcal{E}$ be chosen large enough to guarantee that (a) and (b) hold. Since $D(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, the implicit function theorem (see, for instance, [19, Proposition 6.1, p. 138]) implies that there exists a function $\varphi(\boldsymbol{z})$ that is analytic on a neighborhood of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, say $\mathcal{V}_0$, and such that $P(\boldsymbol{z}, \varphi(\boldsymbol{z})) = 0$. Note that there is a $\overline{\mathbb{Q}}(\boldsymbol{z})$-isomorphism between the field $\overline{\mathbb{Q}}(\boldsymbol{z}, \phi)$ and $\overline{\mathbb{Q}}(\boldsymbol{z}, \varphi(\boldsymbol{z}))$. We thus deduce that the matrix $\boldsymbol{\phi}(\boldsymbol{z}) := \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(\boldsymbol{z})\varphi(\boldsymbol{z})^j$ satisfies the properties of Lemmas 7.4 and 7.6. Furthermore, as $\det \boldsymbol{\phi} \neq 0$, we also deduce that $\det \boldsymbol{\phi}(\boldsymbol{z}) \neq 0$. Hence $\boldsymbol{\phi}(\boldsymbol{z}) \in \mathrm{GL}_t(\mathcal{M}er(\mathcal{V}_0))$. Finally, we deduce that $\det \boldsymbol{\phi}(\boldsymbol{z})$ also satisfies Equation (7.11). Since $q_0(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, we get that $\det \boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$. Hence the matrix $\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})$ is invertible. Furthermore, since $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, the coordinates of $\boldsymbol{\phi}_j(\boldsymbol{z})$ are analytic on some neighborhood of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, say $\mathcal{V}_1$. Finally, setting $\mathcal{V} = \mathcal{V}_0 \cap \mathcal{V}_1$, we obtain that Properties (a)–(d) are satisfied. □

8. Proof of Theorem 6.2

Let $P \in \overline{\mathbb{Q}}[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$ be defined as in Theorem 6.2. Let d_i denote the total degree of P in the indeterminates $\boldsymbol{X}_i$. Set $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$. We keep on with the notation of the previous section. The monomials $\boldsymbol{X}^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{X}^{\boldsymbol{\mu}_t}$

are precisely those which are of total degree d_i in the indeterminates $\boldsymbol{X}_i$, for every i. Hence we have a decomposition of the form

$$P(\boldsymbol{X}) := \sum_{j=1}^{t} \tau_j \boldsymbol{X}^{\boldsymbol{\mu}_j} ,$$

where $\tau_1, \ldots, \tau_t \in \overline{\mathbb{Q}}$. Set $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_t) \in \overline{\mathbb{Q}}^t$ and let $\boldsymbol{X}^{\boldsymbol{\mu}} \in \overline{\mathbb{Q}}[\boldsymbol{X}]^t$ denote the column vector of indeterminates whose coordinates are the monomials $\boldsymbol{X}^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{X}^{\boldsymbol{\mu}_t}$. We write $P(\boldsymbol{X}) = \boldsymbol{\tau}\boldsymbol{X}^{\boldsymbol{\mu}}$. Given a matrix of indeterminates $\boldsymbol{Y} := (y_{i,j})_{1\leq i,j\leq t}$, we set

$$F(\boldsymbol{Y}, \boldsymbol{z}) := \sum_{i,j} \tau_i y_{i,j} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_j} = \boldsymbol{\tau}\boldsymbol{Y}\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}] , \tag{8.1}$$

where we let $\boldsymbol{f}(z) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}^M$ denote the column vector formed by the functions $f_{1,1}(\boldsymbol{z}_1), \ldots, f_{r,m_r}(\boldsymbol{z}_r)$ and $\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}^t$ denote the column vector formed by the functions $\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_t}$. Note that F is a linear form in $\boldsymbol{Y}$. Evaluating at $(\mathrm{I}_t, \boldsymbol{\alpha})$, we obtain

$$F(\mathrm{I}_t, \boldsymbol{\alpha}) = \sum_{j=1}^{t} \tau_j \boldsymbol{f}(\boldsymbol{\alpha})^{\boldsymbol{\mu}_j} = P(\boldsymbol{f}(\boldsymbol{\alpha})) = 0 . \tag{8.2}$$

Remark 8.1. We have $F(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{f}(\boldsymbol{z})] \subset \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]$. Also, $F(\boldsymbol{Y}, \boldsymbol{z})$ can be seen as an element of $\overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$, as we will sometimes do in what follows.

8.1. **Iterated relations.** Using Equality (6.5) and (7.1), we obtain

$$\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_j} = (A_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z}))^{\boldsymbol{\mu}_j} = \sum_{l=1}^{t} R_{j,l}(A_{\boldsymbol{k}}(\boldsymbol{z}))\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z})^{\boldsymbol{\mu}_l} , \tag{8.3}$$

for every j, $1 \leq j \leq t$. We deduce from (8.3) that

$$\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}} = \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z})^{\boldsymbol{\mu}} , \tag{8.4}$$

for all $\boldsymbol{k} \in \mathbb{N}^r$, where $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ is defined as in (7.4).

For every i, $1 \leq i \leq r$, let $b_i(\boldsymbol{z}_i) \in \overline{\mathbb{Q}}[\boldsymbol{z}_i]$ denote the least common multiple of the denominators of the coordinates of $A_i(\boldsymbol{z}_i)$. Hence the matrix $b_i(\boldsymbol{z}_i)A(\boldsymbol{z}_i)$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}_i]$. For every $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we set

$$b_{\boldsymbol{k}}(\boldsymbol{z}) := \prod_{i=1}^{r} \prod_{j=0}^{k_i - 1} b_i(T_i^j \boldsymbol{z}_i)^{d_i} ,$$

so that the matrix $b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}]$. For all $\boldsymbol{k} \in \mathbb{N}^r$, Equality (8.4) implies the following equality in $\overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]$:

$$\begin{aligned} F(\boldsymbol{Y}b_{\boldsymbol{k}}(\boldsymbol{z}), \boldsymbol{z}) &= \boldsymbol{\tau}\boldsymbol{Y}b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}} \\ &= \boldsymbol{\tau}\boldsymbol{Y}b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z})^{\boldsymbol{\mu}} \\ &= F(\boldsymbol{Y}b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) . \end{aligned} \tag{8.5}$$

Every point $\boldsymbol{\alpha}_i$ being regular with respect to the system (6.1.i), the number $b_{\boldsymbol{k}}(\boldsymbol{\alpha})$ is nonzero for all $\boldsymbol{k} \in \mathbb{N}^r$. From (8.2) and the fact that F is linear in $\boldsymbol{Y}$, we deduce that

$$F(\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}}\boldsymbol{\alpha}) = 0 , \ \forall \boldsymbol{k} \in \mathbb{N}^r . \tag{8.6}$$

8.2. **The matrix $\boldsymbol{\Theta}_l(\boldsymbol{z})$.** From now on, we fix a nonnegative integer l_0 and a relation matrix $\boldsymbol{\phi}(\boldsymbol{z})$ satisfying the properties of Lemma 7.7. Set

$$\boldsymbol{\xi} := T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}\,.$$

Properties (a) and (b) in Lemma 7.7 ensure the existence of a real number r_1 such that $0 < \|\boldsymbol{\xi}\| < r_1 < 1$ and such that all the power series $f_{1,1}(\boldsymbol{z}),\ldots,f_{r,m_r}(\boldsymbol{z})$ have a radius of convergence larger than r_1. Then, by Properties (c) and (d) in Lemma 7.7, we can choose a real number r_2 satisfying $0 < \|\boldsymbol{\xi}\| + r_2 < r_1$ and such that the coefficients of the matrix $\boldsymbol{\phi}(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. Note that Property (a) in Lemma 7.7 and Lemma 6.3 imply that $\|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\| \leq \|T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}\|$ for $l \geq l_0$. For every $l \geq l_0$, we set

$$\boldsymbol{\Theta}_l(\boldsymbol{z}) := \boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{\phi}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\,. \tag{8.7}$$

By (7.5), we have $\boldsymbol{\Theta}_l(\boldsymbol{\xi}) = \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$, for all $l \geq l_0$.

Remark 8.2. By Lemma 7.7, the coefficients of $\boldsymbol{\Theta}_{l_0}(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. On the other hand, one has

$$\boldsymbol{\Theta}_l(\boldsymbol{z}) = \boldsymbol{\Theta}_{l-1}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_l-\boldsymbol{k}_{l-1}}(T_{\boldsymbol{k}_{l-1}-\boldsymbol{k}_{l_0}}\boldsymbol{z})\,,\ \forall l \geq l_0\,.$$

This implies that, for every $l \geq l_0$, the coefficients of $\boldsymbol{\Theta}_l(\boldsymbol{z})$ are analytic on some neighborhood of $\boldsymbol{\xi}$, that is on some polydisc $\mathcal{D}(\boldsymbol{\xi}, r_l) \subset \mathcal{D}(\boldsymbol{\xi}, r_2)$. Also, the coefficients of $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. In what follows, we will consider the expression $F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$. Formally, it is a polynomial in $f_{1,1}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}),\ldots,f_{r,m_r}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ and the coordinates of $\boldsymbol{\Theta}_l(\boldsymbol{z})$. Note that it also defines an analytic functions on $\mathcal{D}(\boldsymbol{\xi}, r_l) \subset \mathcal{D}(\boldsymbol{\xi}, r_2)$. In addition, $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z})$ is analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$. Indeed, the functions $f_{1,1}(\boldsymbol{z}),\ldots,f_{r,m_r}(\boldsymbol{z})$ are analytic on $\mathcal{D}(\boldsymbol{0}, r_1) \supset \mathcal{D}(\boldsymbol{\xi}, r_2)$, while our choice of l_0 ensures that the coordinates of $\boldsymbol{\Theta}_{l_0}(\boldsymbol{z})$ are analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$.

8.3. **The key lemma.** In this section, we prove the following result from which we will deduce easily Theorem 6.2 in Section 8.4.

Lemma 8.3. *One has* $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z}) = 0$.

Let us first briefly describe the general strategy we use for proving this key lemma. The scheme of the proof is classical and takes its source in the early work of Mahler [42]. It was gradually improved and refined by Kubota [31], Loxton and van der Poorten [39], and Ku. Nishioka [50, 51, 52]. It takes here a more complicate shape, involving the matrices $\boldsymbol{\Theta}_l$. This is due to the fact that we have to deal with a bunch of Mahler systems of the form (6.1.i) without any restriction on the matrices $A_i(\boldsymbol{z}_i)$.

Assuming by contradiction that $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z}) \neq 0$, we construct, for every triple of nonnegative integers (δ_1, δ_2, l), $l \geq l_0$, an auxiliary function of the form

$$E(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{j=0}^{\delta_1} P_j(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})^j\,,$$

where $P_j(\boldsymbol{Y}, \boldsymbol{z})$ is a polynomial of degree at most δ_1 in each indeterminate $y_{i,j}$ and of total degree at most δ_2 in the indeterminates $z_{i,j}$. Recall that

$E(\boldsymbol{\Theta}_l(\boldsymbol{\xi}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi}) = E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ and that

$$F(\boldsymbol{\Theta}_l(\boldsymbol{\xi}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi}) = F(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,.$$

Hence $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = P_0(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$. As our construction ensures that $P_0 \notin \mathcal{I}$, we have $P_0(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0$ for many integers l, and we can use Liouville's inequality to find a lower bound for $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$. On the other hand, we have many choices for the polynomials P_i in the construction of our auxiliary function. This level of freedom is used to show that, for a good choice of such polynomials, the quantity $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ is small enough. More precisely, we obtain a contradiction between the upper and the lower bounds when the parameter δ_1 is sufficiently large, the parameter δ_2 is sufficiently large with respect to δ_1, and the parameter l is sufficiently large with respect to δ_1 and δ_2.

Warning. The auxiliary function $E(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ can be though of as a simultaneous Padé approximant of type I for the first δ_1th powers of $F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$. However, we have to be careful: $F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ it is not necessarily a power series in $\boldsymbol{z}$. It is a linear combination of the power series $f_{1,1}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}), \ldots, f_{r,m_r}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ whose coefficients are only known to be algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$. We only know that it is analytic at $\boldsymbol{\xi}$.

In what follows, we argue by contradiction, assuming that

$$F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z}) \neq 0\,. \tag{8.8}$$

We divide the proof of Lemma 8.3 into four steps.

8.3.1. *First step: construction of the auxiliary function.* Given a formal power series $E = \sum_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}}(\boldsymbol{Y})\boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$ and an integer $p > 0$, we let

$$E_p := \sum_{|\boldsymbol{\lambda}|<p} e_{\boldsymbol{\lambda}}(\boldsymbol{Y})\boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]$$

denote the truncation of E at order p with respect to $\boldsymbol{z}$. We recall that $\mathcal{I}^{\perp}(\delta_1, \delta_2)$ is a complement to $\mathcal{I}(\delta_1, \delta_2)$ in $\overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$.

Lemma 8.4. *Let $\delta_1 \geq 0$ be an integer. For all integers δ_2, $\delta_2 \gg \delta_1$, there exist polynomials $P_i \in \mathcal{I}^{\perp}(\delta_1, \delta_2)$, $0 \leq i \leq \delta_1$, not all zero, such that the formal power series*

$$E'(\boldsymbol{Y}, \boldsymbol{z}) := \sum_{j=0}^{\delta_1} P_j(\boldsymbol{Y}, \boldsymbol{z})F(\boldsymbol{Y}, \boldsymbol{z})^j \in \overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$$

satisfies $E'_p(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = 0$ for all $l \geq l_0$, where $p = \left\lfloor \frac{\delta_1^{1/N}\delta_2}{2^{(t^2+2)/N}} \right\rfloor$.

Proof. Set

$$\mathcal{J}(\delta_1, \delta_2) := \{P \in \overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{Y}] \ : \ P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y}, \boldsymbol{z}) \in \mathcal{I}(\delta_1, \delta_2)\}\,.$$

The $\overline{\mathbb{Q}}$-vector spaces $\mathcal{J}(\delta_1, \delta_2)$ and $\mathcal{I}(\delta_1, \delta_2)$ have same dimension. This follows directly from the fact that the map

$$\begin{array}{ccc} \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2} & \to & \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2} \\ P(\boldsymbol{Y}, \boldsymbol{z}) & \mapsto & P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y}, \boldsymbol{z}) \end{array}$$

is an isomorphism, the matrix $\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}$ being invertible. Furthermore, we have

$$P(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = 0\,,\ \forall P \in \mathcal{J}(\delta_1, \delta_2),\ \forall l \geq l_0\,. \tag{8.9}$$

Indeed, if $P \in \mathcal{J}(\delta_1, \delta_2)$, then $P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y}, \boldsymbol{z}) \in \mathcal{I}(\delta_1, \delta_2)$, and Lemma 7.6 implies that

$$P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{\phi}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) = 0\,,\ \forall \boldsymbol{k} \in V_+^{\perp}\,.$$

For $l \geq l_0$, using the previous equality with $\boldsymbol{k} := \boldsymbol{k}_l - \boldsymbol{k}_{l_0}$, we obtain that

$$P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{\phi}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = 0\,.$$

By (8.7), we thus have $P(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = 0$.

Let p be as in the lemma and let us consider the three $\overline{\mathbb{Q}}$-linear maps:

$$\left\{\begin{array}{c} \prod_{j=0}^{\delta_1} \mathcal{I}^{\perp}(\delta_1, \delta_2) \\ (P_0(\boldsymbol{Y}, \boldsymbol{z}), \ldots, P_{\delta_1}(\boldsymbol{Y}, \boldsymbol{z})) \end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{c} \overline{\mathbb{Q}}[\boldsymbol{Y}]_{2\delta_1}[[\boldsymbol{z}]] \\ E'(\boldsymbol{Y}, \boldsymbol{z}) := \sum_{j=0}^{\delta_1} P_j(\boldsymbol{Y}, \boldsymbol{z}) F(\boldsymbol{Y}, \boldsymbol{z})^j \end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{c} \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{2\delta_1, p-1} \\ E'_p(\boldsymbol{Y}, \boldsymbol{z}) \end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{l} \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{2\delta_1, p-1}/\mathcal{J}(2\delta_1, p-1) \\ E'_p(\boldsymbol{Y}, \boldsymbol{z}) \mod \mathcal{J}(2\delta_1, p-1) \end{array}\right.$$

Note that these maps are well-defined. By Lemma 7.1, the dimension of the $\overline{\mathbb{Q}}$-vector space $\mathcal{I}^{\perp}(\delta_1, \delta_2)$ is at least equal to $\frac{c_1(\delta_1)}{2}\delta_2^N$, assuming that δ_2 is large enough. It follows that

$$\dim_{\overline{\mathbb{Q}}}\left(\prod_{j=0}^{\delta_1} \mathcal{I}^{\perp}(\delta_1, \delta_2)\right) \geq \frac{c_1(\delta_1)}{2}(\delta_1 + 1)\delta_2^N\,. \tag{8.10}$$

For every pair of nonnegative integers (n, m), set

$$\overline{\mathcal{J}}(n, m) := \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{n,m}/\mathcal{J}(n, m)\,.$$

Since $\mathcal{J}(\delta_1, \delta_2)$ and $\mathcal{I}(\delta_1, \delta_2)$ have same dimension, Lemma 7.2 implies that

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathcal{J}}(2\delta_1, p-1) \leq 2^{t^2} \dim_{\overline{\mathbb{Q}}} \overline{\mathcal{J}}(\delta_1, p-1)\,.$$

Now, if δ_2 is sufficiently large, Lemma 7.1 ensures that

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathcal{J}}(\delta_1, p-1) \leq 2c_1(\delta_1)p^N\,.$$

On the other hand, the choice of p ensures that

$$2^{t^2}\left(2c_1(\delta_1)p^N\right) < \frac{c_1(\delta_1)}{2}(\delta_1 + 1)\delta_2^N$$

and (8.10) implies that

$$\dim_{\overline{\mathbb{Q}}}\left(\prod_{j=0}^{\delta_1}\mathcal{I}^{\perp}(\delta_1,\delta_2)\right) > \dim_{\overline{\mathbb{Q}}}\left(\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,p-1}/\mathcal{J}(2\delta_1,p-1)\right).$$

Hence the $\overline{\mathbb{Q}}$-linear map defined by

$$(P_0(\boldsymbol{Y},\boldsymbol{z}),\ldots,P_{\delta_1}(\boldsymbol{Y},\boldsymbol{z})) \mapsto E'_p(\boldsymbol{Y},\boldsymbol{z}) \mod \mathcal{J}(2\delta_1,p-1)$$

has a nontrivial kernel. We deduce the existence of polynomials $P_0,\ldots,P_{\delta_1}$ in $\mathcal{I}^{\perp}(\delta_1,\delta_2)$, not all zero, such that $E'_p \in \mathcal{J}(2\delta_1,p-1)$. By (8.9), we obtain that $E'_p(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0$ for all $l\geq l_0$. This ends the proof. ☐

Let E' be a formal power series satisfying the properties of Lemma 8.4 and let v_0 be the smallest index such that the polynomial P_{v_0} is nonzero. Then the formal power series

$$E(\boldsymbol{Y},\boldsymbol{z}) := \sum_{j\geq v_0} P_j(\boldsymbol{Y},\boldsymbol{z})F(\boldsymbol{Y},\boldsymbol{z})^{j-v_0} \in \overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]] \tag{8.11}$$

is the auxiliary function that we were looking for. Note that we have

$$E(\boldsymbol{Y},\boldsymbol{z})F^{v_0}(\boldsymbol{Y},\boldsymbol{z}) = E'(\boldsymbol{Y},\boldsymbol{z}). \tag{8.12}$$

8.3.2. *Second step: upper bound for* $|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})|$. The aim of this section is to prove that there exists a real number $c_2>0$ such that

$$|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \leq e^{-c_2e^l\delta_1^{1/N}\delta_2}, \ \ \forall l\gg\delta_2\gg\delta_1. \tag{8.13}$$

Let us first observe that

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E'(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$$

defines an analytic function on $\mathcal{D}(\boldsymbol{\xi},r_2)$. Then, on this polydisc, it has a power series expansion

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E'(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N}\epsilon_{\boldsymbol{\lambda},l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}, \tag{8.14}$$

where $\epsilon_{\boldsymbol{\lambda},l}\in\mathbb{C}$. Let p be defined as in Lemma 8.4. We first prove the following result.

Lemma 8.5. *There exists a positive real number γ that does not depend on δ_1,δ_2, $\boldsymbol{\lambda}$, and l, such that*

$$|\epsilon_{\boldsymbol{\lambda},l}| \leq e^{-\gamma e^l p}, \ \forall l\gg\delta_1,\delta_2,\boldsymbol{\lambda}.$$

Proof. Set

$$G(\boldsymbol{Y},\boldsymbol{z}) := E'(\boldsymbol{Y},\boldsymbol{z}) - E'_p(\boldsymbol{Y},\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}].$$

By Lemma 8.4, we have

$$G(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = E'(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}), \ \forall l\geq l_0. \tag{8.15}$$

Let $\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_s$ denote an enumeration of the $t\times t$ matrices with nonnegative integer coefficients that can be decomposed as

$$\boldsymbol{\nu}_l = M_l + N_l,$$

where M_l has coefficients in the set $\{0,1,\ldots,\delta_1\}$ and N_l is a matrix with nonnegative integer coefficients whose sum is at most δ_1. The sum of the coefficients of each matrix $\boldsymbol{\nu}_i$ is at most equal to $(t^2+1)\delta_1$. There exists a unique decomposition of the form

$$G(\boldsymbol{Y},\boldsymbol{z}) = \sum_{i=1}^{s} \sum_{|\boldsymbol{\lambda}|\geq p} g_{\boldsymbol{\lambda},i}\boldsymbol{z}^{\boldsymbol{\lambda}}\boldsymbol{Y}^{\boldsymbol{\nu}_i}\,,$$

where $g_{\boldsymbol{\lambda},i} \in \mathbb{C}$. For every i, $1 \leq i \leq s$, we define the formal power series

$$G_i(\boldsymbol{z}) := \sum_{|\boldsymbol{\lambda}|\geq p} g_{\boldsymbol{\lambda},i}\boldsymbol{z}^{\boldsymbol{\lambda}} \in \mathbb{C}[[\boldsymbol{z}]]\,.$$

By definition of $F(\boldsymbol{Y},\boldsymbol{z})$, these series belong to $\overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z})]$. In particular, they are analytic on some polydisc $\mathcal{D}(\mathbf{0},r)$ with $r > r_1$. By (3.1), there exists a positive real number $\gamma_1(\delta_1,\delta_2)$ such that

$$|g_{\boldsymbol{\lambda},i}| \leq \gamma_1(\delta_1,\delta_2)r_1^{-|\boldsymbol{\lambda}|}\,,\ \forall \boldsymbol{\lambda}\in\mathbb{N}^N\,. \tag{8.16}$$

On the other hand, Lemma 5.5 ensures the existence of two positive real numbers κ_1 and κ_2 such that

$$\kappa_1 e^l|\boldsymbol{\lambda}| \leq |\boldsymbol{\lambda}T_{\boldsymbol{k}_l}| \leq \kappa_2 e^l|\boldsymbol{\lambda}| \quad \forall l, \forall \boldsymbol{\lambda}. \tag{8.17}$$

For every $l \geq l_0$, $G_i(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ can thus be written as

$$G_i(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{|\boldsymbol{\lambda}|\geq\kappa_1 e^l p} g_{\boldsymbol{\lambda},i,l}\boldsymbol{z}^{\boldsymbol{\lambda}}\,, \tag{8.18}$$

with $g_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$. Furthermore, this power series is absolutely convergent on the polydisc $\mathcal{D}(\mathbf{0},r_1)$. Since the matrix $T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}$ is invertible, has nonnegative coefficients, and $r_1 \leq 1$, we deduce from (8.16) that

$$|g_{\boldsymbol{\lambda},i,l}| \leq \gamma_1(\delta_1,\delta_2)r_1^{-|\boldsymbol{\lambda}|}\,, \tag{8.19}$$

for all $\boldsymbol{\lambda}$, $i \in \{1,\ldots,s\}$, and $l \geq l_0$. On the other hand, every function $G_i(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$, $1 \leq i \leq s$, $l \geq l_0$, is analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi},r_2)$. The power series expansion of $G_i(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ at $\boldsymbol{\xi}$ is thus absolutely convergent on $\mathcal{D}(\boldsymbol{\xi},r_2)$, and we can write

$$G_i(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} h_{\boldsymbol{\lambda},i,l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{8.20}$$

where $h_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$. Since by assumption $\mathcal{D}(\boldsymbol{\xi},r_2) \subset \mathcal{D}(\mathbf{0},r_1)$, the two power series expansions (8.18) and (8.20) match on $\mathcal{D}(\boldsymbol{\xi},r_2)$. Using the equality $\boldsymbol{z}^{\boldsymbol{\lambda}} = ((\boldsymbol{z}-\boldsymbol{\xi})+\boldsymbol{\xi})^{\boldsymbol{\lambda}}$ and identifying, for every $\boldsymbol{\lambda}\in\mathbb{N}^N$, the terms in $(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}$ in (8.18) and (8.20), we obtain that

$$h_{\boldsymbol{\lambda},i,l} = \sum_{\substack{|\boldsymbol{\gamma}|\geq\kappa_1 e^l p\\ \boldsymbol{\gamma}\geq\boldsymbol{\lambda}}} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} g_{\boldsymbol{\gamma},i,l}\boldsymbol{\xi}^{\boldsymbol{\gamma}-\boldsymbol{\lambda}}\,. \tag{8.21}$$

For $\boldsymbol{\gamma}\geq\boldsymbol{\lambda}$, one has

$$\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} = \prod_{i=1}^{N} \frac{\gamma_i!}{(\gamma_i-\lambda_i)!\lambda_i!} \leq \prod_{i=1}^{N}\gamma_i^{\lambda_i} \leq |\boldsymbol{\gamma}|^{|\boldsymbol{\lambda}|}\,. \tag{8.22}$$

Given $\boldsymbol{\lambda} \in \mathbb{N}^N$, $|\boldsymbol{\lambda}| < \kappa_1 e^l p$ as soon as l is large enough, and since $\|\boldsymbol{\xi}\| < r_1$, we infer from (8.19) and (8.21) the existence of a real number $\gamma_2 > 0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l, such that

$$|h_{\boldsymbol{\lambda},i,l}| \leq e^{-\gamma_2 e^l p}\,,\ \forall l \gg \delta_1, \delta_2, \boldsymbol{\lambda}\,. \tag{8.23}$$

By Remark 8.2, the monomial $(b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z}))^{\boldsymbol{\nu}_i}$ is analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$ for every i, $1 \leq i \leq s$, and every $l \geq l_0$. Multiplying by $b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1 - |\boldsymbol{\nu}_i|}$, we obtain on $\mathcal{D}(\boldsymbol{\xi}, r_2)$ a power series expansion of the form

$$b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1 - |\boldsymbol{\nu}_i|}(b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z}))^{\boldsymbol{\nu}_i} = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \theta_{\boldsymbol{\lambda},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{8.24}$$

where $\theta_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$.

Given $Q(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ and $l \geq 0$, the polynomial $Q_l(\boldsymbol{z}) := Q(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}\boldsymbol{z})$ can be converted into a polynomial in $(\boldsymbol{z} - \boldsymbol{\xi})$:

$$Q_l(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}} q_{\boldsymbol{\lambda},l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,.$$

Using the relation

$$\boldsymbol{z}^{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\lambda} \leq \boldsymbol{\gamma}} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} \boldsymbol{\xi}^{\boldsymbol{\lambda} - \boldsymbol{\gamma}}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,,$$

(8.17) and (8.22), we obtain that, for every $\boldsymbol{\lambda}$, $|q_{\boldsymbol{\lambda},l}| = \mathcal{O}(e^l)$, where the underlying constant in the $\mathcal{O}$ notation depends both on $Q(\boldsymbol{z})$ and $\boldsymbol{\lambda}$. Set $\Gamma_{\boldsymbol{k}}(\boldsymbol{z}) := b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ and

$$\Gamma_{\boldsymbol{k},l}(\boldsymbol{z}) := b_{\boldsymbol{k}}(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}\boldsymbol{z})\,.$$

Since there are only finitely many differences $\boldsymbol{k}_{l+1} - \boldsymbol{k}_l$, the coefficients in $(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}$ for the entries of $\Gamma_{\boldsymbol{k}_{l+1} - \boldsymbol{k}_l, l}(\boldsymbol{z})$, seen as polynomials in $\boldsymbol{z} - \boldsymbol{\xi}$, belong to $\mathcal{O}(e^l)$. Using the recurrence relation

$$b_{\boldsymbol{k}_{l+1} - \boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_{l+1}(\boldsymbol{z}) = b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z})\Gamma_{\boldsymbol{k}_{l+1} - \boldsymbol{k}_l, l}(\boldsymbol{z})\,,$$

we infer from (6.6) that, for every $\boldsymbol{\lambda}$, there exists a real number $\gamma_3(\delta_1, \boldsymbol{\lambda}) > 0$ that does not depend on i, δ_2, and l, and such that

$$|\theta_{\boldsymbol{\lambda},i,l}| < e^{\gamma_3(\delta_1, \boldsymbol{\lambda})l}\,,\ \ \forall i,\ 1 \leq i \leq s,\ \forall l \geq l_0\,. \tag{8.25}$$

From (8.14), (8.15), (8.20), and (8.24), we deduce that

$$\sum_{i=1}^{s}\left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} h_{\boldsymbol{\lambda},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\right)\left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \theta_{\boldsymbol{\lambda},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\right) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \epsilon_{\boldsymbol{\lambda},l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,. \tag{8.26}$$

Moreover, since the power series involved in (8.20) and (8.24) are absolutely convergent on $\mathcal{D}(\boldsymbol{\xi}, r_2)$, we obtain

$$\sum_{i=1}^{s}\left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} h_{\boldsymbol{\lambda},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\right)\left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \theta_{\boldsymbol{\lambda},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\right)$$

$$= \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \sum_{i=1}^{s} \sum_{\boldsymbol{\gamma} \leq \boldsymbol{\lambda}} h_{\boldsymbol{\gamma},i,l}\theta_{\boldsymbol{\lambda} - \boldsymbol{\gamma},i,l}(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,. \tag{8.27}$$

Finally, identifying the terms in $(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}$ in (8.26) thanks to (8.27), we get that

$$\epsilon_{\boldsymbol{\lambda},l} = \sum_{i=1}^{s}\sum_{\boldsymbol{\gamma}\leq\boldsymbol{\lambda}} h_{\boldsymbol{\gamma},i,l}\theta_{\boldsymbol{\lambda}-\boldsymbol{\gamma},i,l}\,.$$

Now, we fix $\boldsymbol{\lambda}\in\mathbb{N}^N$. By (8.23) and (8.25), there exists a real number $\gamma_4>0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l, such that

$$|\epsilon_{\boldsymbol{\lambda},l}| \leq e^{-\gamma_4 e^l p}\,, \quad \forall l \gg \delta_1,\delta_2,\boldsymbol{\lambda}\,.$$

Setting $\gamma=\gamma_4$, this ends the proof. $\square$

Now, let us observe that the function $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ is analytic on $\mathcal{D}(\boldsymbol{\xi},\boldsymbol{r}_2)$. Hence it has a power series expansion of the form

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} e_{\boldsymbol{\lambda},l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{8.28}$$

with $e_{\boldsymbol{\lambda},l}\in\mathbb{C}$. Also, $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0}$ is analytic on $\mathcal{D}(\boldsymbol{\xi},r_2)$, so that

$$F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0} = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} a_{\boldsymbol{\lambda}}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{8.29}$$

with $a_{\boldsymbol{\lambda}}\in\mathbb{C}$. Using (8.5), we get that

$$F(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})\,, \quad \forall l\geq l_0\,.$$

This equality is *a priori* valid for $\boldsymbol{z}$ in $\mathcal{D}(\boldsymbol{\xi},r_l)$ (see Remark 8.2), but it extends to $\mathcal{D}(\boldsymbol{\xi},r_2)$ by analytic continuation. By (8.12), we thus have

$$E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0} = E'(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})\,, \tag{8.30}$$

for all $l\geq l_0$ and all $\boldsymbol{z}\in\mathcal{D}(\boldsymbol{\xi},r_l)$. By (8.8), $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})$ is nonzero and there thus exists at least one nonzero coefficient $a_{\boldsymbol{\lambda}}$ in (8.29). Let us consider an index $\boldsymbol{\lambda}_0$ such that $a_{\boldsymbol{\lambda}_0}\neq 0$ with $|\boldsymbol{\lambda}_0|$ minimal. Multiplying both sides of (8.30) by $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}$, and identifying the coefficients in $(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}_0}$ in their power series expansion on $\mathcal{D}(\boldsymbol{\xi},r_2)$ with the help of (8.14), (8.28), and (8.29), we obtain that

$$e_{0,l}a_{\boldsymbol{\lambda}_0} = \epsilon_{\boldsymbol{\lambda}_0,l}\,, \quad \forall l\geq l_0\,.$$

Since $\boldsymbol{\Theta}_l(\boldsymbol{\xi}) = \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$, we infer from Lemma 8.5 and the definition of p (see Lemma 8.4), the existence of a real number $\beta_1>0$ that does not depend on δ_1, δ_2, and l, such that

$$\begin{aligned} & \left|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})\right| \\ = & \left|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{\xi}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi})\right| \\ = & \ |e_{0,l}| \\ \leq & \ e^{-\beta_1 e^l\delta_1^{1/N}\delta_2}\,, \quad \forall l\gg\delta_1,\delta_2\,. \end{aligned} \tag{8.31}$$

Now, it just remains to find a lower bound for $|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}|$. By (8.17), there exists a positive real number β_2 that does not depend on l, δ_1, and δ_2, such that the degree of the polynomial $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}$ is at most equal to $\beta_2 e^l\delta_1$. Each $\boldsymbol{\alpha}_i$ being by assumption regular w.r.t. (6.1.i), $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})\neq 0$ for all $l\geq l_0$. Since the numbers $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})$, $l\geq l_0$, belong to some fix number field, we infer from Liouville's inequality the existence of a real number $\beta_3 > 0$ that does not depend on l, δ_1, and δ_2, such that

$$|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})|^{(t^2+1)\delta_1} \geq e^{-\beta_3 e^l \delta_1}\,, \quad \forall l,\, \forall \delta_1\,. \tag{8.32}$$

By (8.31) and (8.32), there exists a real number $\beta_4 > 0$ that does not depend on l, δ_1, and δ_2, such that

$$\begin{aligned} |E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| &\leq e^{\beta_3 e^l \delta_1 - \beta_1 e^l \delta_1^{1/N}\delta_2} \\ &\leq e^{-\beta_4 e^l \delta_1^{1/N}\delta_2}\,, \quad \forall l \gg \delta_2 \gg \delta_1\,. \end{aligned}$$

Setting $c_2 = \beta_4$, this proves the upper bound (8.13).

8.3.3. *Third step: lower bound for* $|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})|$. Let us first recall that, by (8.6), we have

$$F(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,, \quad \forall l\,.$$

By construction of our auxiliary function, we deduce that

$$E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})\,.$$

Furthermore, since this construction ensures that $P_{v_0} \notin \mathcal{I}$, there exists an infinite set of positive integers $\mathcal{E}$ such that

$$P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0\,, \quad \forall l \in \mathcal{E}\,.$$

Since P_{v_0} has degree at most δ_1 in each indeterminate $y_{i,j}$, and total degree at most δ_2 in $\boldsymbol{z}$, Liouville's inequality and a computation similar to the previous one ensure the existence of a real number $c_3 > 0$ that does not depend on δ_1, δ_2, and l, such that

$$|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| = |P_0(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \geq e^{-c_3 e^l \delta_2}\,, \quad \forall l \in \mathcal{E}, \delta_2 \geq \delta_1\,. \tag{8.33}$$

8.3.4. *Fourth step: contradiction.* By Inequalities (8.13) and (8.33), we obtain that

$$e^{-c_3 e^l \delta_2} \leq |E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \leq e^{-c_2 e^l \delta_1^{1/N}\delta_2}\,, \quad \forall l \in \mathcal{E},\, l \gg \delta_2 \gg \delta_1\,.$$

Finally, we deduce that

$$c_3 \geq c_2 \delta_1^{1/N}\,.$$

Since c_2 and c_3 do not depend on δ_1, this inequality provides a contradiction, as soon as δ_1 is large enough. This end the proof of Lemma 8.3. □

8.4. **End of the proof of Theorem 6.2.** Let us recall that $d(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ stands for the least common multiple of the denominators of the coefficients of the matrices $\boldsymbol{\phi}_j(\boldsymbol{z})$ defined in (7.10). Hence $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}, \varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})]$. Lemma 7.7 ensures that $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$. Let $q(\boldsymbol{z})$ denote the least common multiple of the denominators of the coefficients of the matrix $\boldsymbol{R}_{\boldsymbol{k}_{l_0}}^{-1}(\boldsymbol{z})$. Since, for every i, $\boldsymbol{\alpha}_i$ is assumed to be regular w.r.t. (6.1.i), we have that $q(\boldsymbol{\alpha}) \neq 0$. Similarly, $b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}]$ and $b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha}) \neq 0$.

By Lemma 8.3, we know that $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})=0$, and substituting $T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}$ to $\boldsymbol{z}$, we obtain that $F(\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}),T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})=0$. The function $F(\boldsymbol{Y},\boldsymbol{z})$ being linear in $\boldsymbol{Y}$, we deduce that

$$F\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}),T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}\right)=0\,.$$

Writing $\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})=\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1}\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})$, and using (8.5), we get that

$$F\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1},\boldsymbol{z}\right)=0\,. \tag{8.34}$$

Set

$$Q(\boldsymbol{z},\boldsymbol{X}):=\boldsymbol{\tau}\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1}\right)\boldsymbol{X}^{\boldsymbol{\mu}}\,,$$

where we let $\boldsymbol{X}^{\boldsymbol{\mu}}$ denote the column vector with coordinates $\boldsymbol{X}^{\boldsymbol{\mu}_1},\ldots,\boldsymbol{X}^{\boldsymbol{\mu}_t}$. We thus get that $Q(\boldsymbol{z},\boldsymbol{X})\in\overline{\mathbb{Q}}[\boldsymbol{z},\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}),\boldsymbol{X}]$. Since $\varphi\circ T_{\boldsymbol{k}_{l_0}}$ is analytic at $\boldsymbol{\alpha}$, it follows that $Q(\boldsymbol{z},\boldsymbol{X})\in\overline{Q(\boldsymbol{z})}_{\boldsymbol{\alpha}}[\boldsymbol{X}]$. Moreover, since $\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})=\boldsymbol{\Theta}_{l_0}(\boldsymbol{\xi})=\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})$, we deduce that

$$Q(\boldsymbol{\alpha},\boldsymbol{X})=\boldsymbol{\tau}\boldsymbol{X}^{\boldsymbol{\mu}}=P(\boldsymbol{X})\,.$$

Finally, it follows from (8.34) that

$$Q(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z}))=0\,.$$

This ends the first part of the proof of Theorem 6.2.

Now, let us assume that $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let $\mathbb{K}$ be an algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ containing $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$. By [33, Chapter VIII], $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$ and $\mathbb{K}$ are linearly disjoint over $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let δ denote the degree of $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$ over $\overline{\mathbb{Q}}(\boldsymbol{z})$. Since the functions $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})^j$, $0\le j\le\delta-1$, are linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})$, they remain linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$. Splitting the polynomial Q as

$$Q=\sum_{j=0}^{\delta-1}Q_j(\boldsymbol{z},\boldsymbol{X})\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})^j\,,$$

where $Q_j(\boldsymbol{z},\boldsymbol{X})\in\overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{X}]$, we thus deduce that $Q_j(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z}))=0$ for all j, $0\le j\le\delta-1$. Finally, setting

$$R(\boldsymbol{z},\boldsymbol{X}):=\sum_{j=0}^{\delta-1}Q_j(\boldsymbol{z},\boldsymbol{X})\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^j\in\overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{X}]\,,$$

we obtain that $R(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z}))=0$ and $R(\boldsymbol{\alpha},\boldsymbol{X})=P(\boldsymbol{X})$, as wanted. □

9. Proofs of Theorems 2.3, 2.6, 2.8, and of Corollaries 2.5 and 2.9

In this section, we complete the proof of our main results. Note that we establish Corollary 2.9 before Theorems 2.6 and 2.8. The latter are in fact deduced from Corollary 2.9.

9.1. **Proof of Theorem 2.3.** There is nothing more to do. Theorem 2.3 just corresponds to the case $r = 1$ of Theorem 6.2. $\square$

9.2. **Proof of Corollary 2.5.** We first note that the inequality

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) \leq \mathrm{tr.deg}_{\overline{\mathbb{Q}(\boldsymbol{z})}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))$$

always holds. Let $\mathbb{K}$ denote the field of fractions of the ring $\overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$. Since $\mathbb{K}$ is algebraic over $\overline{\mathbb{Q}}(z)$, we have

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}(\boldsymbol{z})}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) = \mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))\,.$$

It thus remains to prove that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) \geq \mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))\,. \tag{9.1}$$

We follow the same strategy as the one used for proving the Siegel–Shidlovskii theorem in the framework of E-functions (see [25, Theorem 5.23, p. 230]). We first replace Proposition 5.1 in [25] by the following result.

Lemma 9.1. *Let $g_1(\boldsymbol{z}), \ldots, g_\ell(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be related by a linear T-Mahler system. Let $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ be a regular point w.r.t. this system and let us assume that the pair $(T, \boldsymbol{\alpha})$ is admissible. Let s be the maximum number of functions among $g_1(\boldsymbol{z}), \ldots, g_\ell(\boldsymbol{z})$ that are linearly independent over $\mathbb{K}$. Then at least s of the numbers $g_1(\boldsymbol{\alpha}), \ldots, g_\ell(\boldsymbol{\alpha})$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. Let t denote the dimension of the $\overline{\mathbb{Q}}$-vector space spanned by the numbers $g_i(\boldsymbol{\alpha})$, $1 \leq i \leq \ell$. Without any loss of generality, we assume that $g_1(\boldsymbol{\alpha}), \ldots, g_t(\boldsymbol{\alpha})$ are linearly independent over $\overline{\mathbb{Q}}$. Then, for every i, $t < i \leq \ell$, there exist algebraic numbers $\gamma_{i,1}, \ldots, \gamma_{i,t}$ such that

$$g_i(\boldsymbol{\alpha}) = \gamma_{i,1} g_1(\boldsymbol{\alpha}) + \cdots + \gamma_{i,t} g_t(\boldsymbol{\alpha})\,.$$

By Theorem 2.3, there exist $p_{i,1}(\boldsymbol{z}), \ldots, p_{i,\ell}(\boldsymbol{z}) \in \overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$ such that

$$p_{i,1}(\boldsymbol{z}) g_1(\boldsymbol{z}) + \cdots + p_{i,\ell}(\boldsymbol{z}) g_\ell(\boldsymbol{z}) = 0\,,$$

with $p_{i,i}(\boldsymbol{\alpha}) = 1$ and $p_{i,j}(\boldsymbol{\alpha}) = 0$ when $t < j \leq \ell$ and $j \neq i$. Set

$$L_i(\boldsymbol{z}, X_1, \ldots, X_\ell) := p_{i,1}(\boldsymbol{z}) X_1 + \cdots + p_{i,\ell}(\boldsymbol{z}) X_\ell = 0\,,\ t < i \leq \ell\,,.$$

Note that the linear form $L_i(\boldsymbol{\alpha}, X_1, \ldots, X_m)$ is equal to

$$X_i + \sum_{j=1}^{t} \gamma_{i,j} X_j\,.$$

Hence the linear forms $L_i(\boldsymbol{\alpha}, X_1, \ldots, X_m)$, $t < i \leq \ell$, are linearly independent over $\overline{\mathbb{Q}}$ and *a fortiori* the corresponding linear forms $L_i(\boldsymbol{z}, X_1, \ldots, X_\ell)$ are linearly independent over $\mathbb{K}$. It follows that $t \geq s$, as wanted. $\square$

Proof of Corollary 2.5. Let $D \geq 0$ be an integer. As in [25, p. 231], we let $\varphi_{\boldsymbol{\alpha}}(D)$ denote the dimension of the $\overline{\mathbb{Q}}$-vector space spanned by the monomials of total degree at most D in $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$. We also let $\varphi_{\boldsymbol{z}}(D)$ denote the dimension of the $\mathbb{K}$-vector space spanned by the monomials of total degree at most D in $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$. Now, we observe that the monomials of total degree at most D in $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ are also related by a linear T-Mahler system for which $\boldsymbol{\alpha}$ remains regular. Indeed, such a system can be obtained by first taking the system associated with the matrix $((1) \oplus A(\boldsymbol{z}))^{\otimes d}$, that is, the Dth power of Kronecker of the matrix $(1) \oplus A(\boldsymbol{z})$, where $A(\boldsymbol{z})$ is defined as in (2.2), and then applying some standard reduction procedure (see [24, Lemme 5]). Then, we infer from Lemma 9.1 that

$$\varphi_{\boldsymbol{\alpha}}(D) \geq \varphi_{\boldsymbol{z}}(D),\ \forall D \in \mathbb{N}\,. \tag{9.2}$$

By the Hilbert-Serre Theorem (see for example [56, Theorem 42, p. 235]), for sufficiently large D, $\varphi_{\boldsymbol{\alpha}}(D)$ and $\varphi_{\boldsymbol{z}}(D)$ are polynomials in D whose degree are respectively $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha}))$ and $\operatorname{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))$. Hence (9.2) implies that Inequality (9.1) holds, as wanted. □

9.3. **Proof of Corollary 2.9.** We first need the two following simple results.

Lemma 9.2. *Let $\boldsymbol{z} = (z_{1,1},\ldots,z_{1,n_1},z_{2,1},\ldots,z_{r,n_r})$ be a tuple of $n_1+\cdots+n_r$ distinct variables. For every i, $1 \leq i \leq r$, let $f_{i,1}(\boldsymbol{z}_i),\ldots,f_{i,m_i}(\boldsymbol{z}_i) \in \overline{\mathbb{Q}}[[\boldsymbol{z}_i]]$ be some power series, where $\boldsymbol{z}_i = (z_{i,1},\ldots,z_{i,n_i})$. Then*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq i \leq r,\ 1 \leq j \leq m_i\} = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_i)}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq j \leq m_i\}\,.$$

Proof. The result follows directly from the fact that the sets of variables $\{z_{1,1},\ldots,z_{1,n_1}\},\ldots,\{z_{r,1},\ldots,z_{r,n_r}\}$ are disjoint. □

Lemma 9.3. *Let $\mathcal{E}_1,\ldots,\mathcal{E}_r,\mathcal{F}_1,\ldots,\mathcal{F}_r$ be nonempty finite sets of complex numbers such that $\mathcal{E}_i \subset \mathcal{F}_i$ for every i, $1 \leq i \leq r$. Let us assume that*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{F}_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_i)\,.$$

Then

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,.$$

Proof. Suppose first that all elements of each $\mathcal{F}_i$, $1 \leq i \leq r$, are algebraically independent. By assumption, all elements of the set $\bigcup_{i=1}^{r} \mathcal{F}_i$ are algebraically independent. Hence, all elements of the set $\bigcup_{i=1}^{r} \mathcal{E}_i$ are also algebraically independent, and the lemma is proved. Let us assume now that some elements of the family $\mathcal{F}_i$, $1 \leq i \leq r$, are algebraically dependent. For every i, we choose a subset of algebraically independent elements $\mathcal{E}'_i \subset \mathcal{E}_i$ such that $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}'_i) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$. Then, we complete the set $\mathcal{E}'_i$ in a set of algebraically independent elements $\mathcal{F}'_i \subset \mathcal{F}_i$ such that $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}'_i) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_i)$. From the first part of the proof, we have

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}'_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}'_i)\,.$$

It follows that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i\right) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}'_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}'_i) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,,$$

which ends the proof. □

We are now ready to prove Corollary 2.9.

Proof of Corollary 2.9. We continue with the notation of Theorem 2.6 and 2.8.

Let us first assume that the assumptions of Theorem 2.6 are satisfied. We can gather all the linear Mahler systems (2.8.i) into a big Mahler system of the form (2.2), where $A(\boldsymbol{z}) = A_1(\boldsymbol{z}_1) \oplus \cdots \oplus A_r(\boldsymbol{z}_r)$, $\boldsymbol{z} = (z_{1,1}, \ldots, z_{r,m_r})$, and $T := T_1 \oplus \cdots \oplus T_r$. Then, we infer from assumptions (i) and (ii) of Theorem 2.6 and from Theorem 4.6 that the pair $(T, \boldsymbol{\alpha})$ is admissible and that the point $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$ is regular with respect to this T-Mahler system. Hence we can apply Corollary 2.5 to this larger system. We obtain that

$$\begin{aligned}(9.3)\quad \operatorname{tr.deg}_{\overline{\mathbb{Q}}}&\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \le i \le r,\ 1 \le j \le m_i\} \\ &= \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\{f_{i,j}(\boldsymbol{z}_i) : 1 \le i \le r,\ 1 \le j \le m_i\}\,.\end{aligned}$$

On the other hand, applying Corollary 2.5 to the system (2.8.i), for every i, $1 \le i \le r$, we deduce that

$$(9.4)\quad \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i)\} : 1 \le j \le m_i\} = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_i)}\{f_{i,j}(\boldsymbol{z}_i) : 1 \le j \le m_i\}\,.$$

It follows from (9.3), (9.4), and Lemma 9.2 that

$$\begin{aligned}(9.5)\quad \operatorname{tr.deg}_{\overline{\mathbb{Q}}}&\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \le i \le r,\ 1 \le j \le m_i\} \\ &= \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \le j \le m_i\}\,.\end{aligned}$$

For every i, set $\mathcal{F}_i := \{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \le j \le m_i\}$. Using (9.5), we can thus apply Lemma 9.3 to deduce that $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$, as wanted.

Now, let us assume that the assumptions of Theorem 2.8 are satisfied. The proof is essentially the same. The only change occurs when establishing Equality (9.3). We infer from assumptions (i) and (ii) of Theorem 2.8 that we can apply Theorem 6.2. Then, using Theorem 6.2 and arguing as in the proof of Corollary 2.5, we deduce that Equality (9.3) holds. The rest of the proof remains unchanged. □

9.4. **Proof of Theorems 2.6 and 2.8.** We continue with the notation of Theorem 2.6 and 2.8. For every i, $1 \le i \le r$, we set

$$\mathcal{E}_i := (f_{i,\ell_1}(\boldsymbol{\alpha}_i), \ldots, f_{i,\ell_{s_i}}(\boldsymbol{\alpha}_i))\,,$$

where $1 \le \ell_1 < \ell_2 < \cdots < \ell_{s_i} \le m_i$. Note that the inclusion

$$(9.6)\qquad \sum_{i=1}^{r} \operatorname{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \subset \operatorname{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$$

is trivial. Suppose that the assumptions of either Theorem 2.6 or Theorem 2.8 hold. By Corollary 2.9, we have

$$(9.7)\qquad \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,.$$

Given an ideal $\mathcal{I}$, we let $\operatorname{ht}(\mathcal{I})$ denote its height. Then, we have

$$(9.8)\qquad \begin{aligned}\operatorname{ht}\Big(\operatorname{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\Big) &= s_i - \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,,\ \ \forall i,\ 1 \le i \le r, \\ \operatorname{ht}\Big(\operatorname{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})\Big) &= S - \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E})\,,\end{aligned}$$

where $S := s_1 + \cdots + s_r$. From (9.7) and (9.8) we deduce that

$$\mathrm{ht}\left(\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})\right) = \sum_{i=1}^{r} \mathrm{ht}\left(\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\right) .$$

Set $\mathcal{I} := \sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})$. Then the isomorphism[7]

$$\overline{\mathbb{Q}}[\boldsymbol{X}_1]/\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_1) \otimes_{\overline{\mathbb{Q}}} \cdots \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\boldsymbol{X}_r]/\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_r) \cong \overline{\mathbb{Q}}[\boldsymbol{X}]/\mathcal{I}$$

implies that $\mathcal{I}$ is a prime ideal. Indeed, the tensor product of integral domains, over an algebraically closed field, is an integral domain. Furthermore, this isomorphism also gives that $\mathrm{ht}(\mathcal{I}) = \sum_{i=1}^{r} \mathrm{ht}(\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i))$. It follows that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ and $\sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})$ are both prime ideals with the same height. By (9.6), these two ideals are equal. This ends the proof. $\square$

10. Proof of Theorem 1.1

In this section, we show how to deduce Theorem 1.1 from the two purity theorems. We first prove the following lemma.

Lemma 10.1. *Let $f(z)$ be a q-Mahler function and α be a nonzero algebraic number such that $f(z)$ is well-defined at α. Then there exists a q-Mahler function $g(z)$ such that the following properties hold.*

(a) $g(\alpha) = f(\alpha)$.

(b) *The function $g(z)$ is the first coordinate of a vector solution to a q^l-Mahler system, say*

$$\begin{pmatrix} g_1(z) = g(z) \\ \vdots \\ g_m(z) \end{pmatrix} = B(z) \begin{pmatrix} g_1(z^{q^l}) \\ \vdots \\ g_m(z^{q^l}) \end{pmatrix} .$$

for some integer $l > 0$.

(c) *The point α is regular with respect to this system.*

Proof. We first note that if $f(\alpha)$ is algebraic, the lemma is trivial for we can choose $g(z) := f(\alpha)$ to be constant. We assume now that $f(\alpha)$ is transcendental. Using a minimal q-Mahler equation for $f(z)$, we deduce that $f(z)$ is the first coordinate of some q-Mahler system, say

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^q) \\ \vdots \\ f_m(z^q) \end{pmatrix} , \tag{10.1}$$

where $f_1(z) = f(z), \ldots, f_m(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$. Since the functions $f_1(z), \ldots, f_m(z)$ are linearly independent, we infer from [8, Theorem 1.10] that there exists an integer l such that the numbers α^{q^l} is regular with respect to the system

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A_l(z) \begin{pmatrix} f_1(z^{q^l}) \\ \vdots \\ f_m(z^{q^l}) \end{pmatrix} , \tag{10.2}$$

[7] See, for instance, [29, Chap. I, Exercise 3.15].

where
$$A_l(z) = A(z)A(z^q)\cdots A(z^{q^{l-1}})\,.$$
Furthermore, this theorem ensures that α is not a pole of the matrix $A_l(z)$. Let $(a_1(z),\ldots,a_m(z))$ denote the first row of $A_l(z)$. Set

$$g(z) = a_1(\alpha)f_1(z^{q^l}) + \cdots + a_m(\alpha)f_m(z^{q^l})\,. \tag{10.3}$$

Note that $g(z)$ is a q-Mahler function for it is obtained as a linear combination over $\overline{\mathbb{Q}}$ of q-Mahler functions[8]. Since $f(\alpha)$ is transcendental, the vector $(a_1(\alpha),\ldots,a_m(\alpha))$ is nonzero. Then applying a suitable constant gauge transformation to the Mahler system associated with the matrix $A_l(z^{q^l})$, we can obtain a Mahler system which has a solution vector with $g(z)$ as first coordinate. Furthermore, since the point α^{q^l} is regular w.r.t. (10.2), α is a regular point w.r.t. this new system. On the other hand, we infer from (10.2) and (10.3) that $g(\alpha) = f(\alpha)$, as wanted. $\square$

Proof of Theorem 1.1. We keep on with the notation of Theorem 1.1. We assume that no number among $f_1(\alpha_1),\ldots,f_r(\alpha_r)$ belongs to $\mathbb{K}$, so that it remains to prove that $f_1(\alpha_1),\ldots,f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$. By [8, Corollaire 1.8], our assumption implies that the numbers $f_1(\alpha_1),\ldots,f_r(\alpha_r)$ are all transcendental. For every i, $1 \le i \le r$, we let z_i denote an indeterminate.

By Lemma 10.1, with each pair (f_i,α_i), we can associate a q_i-Mahler function $g_i(z_i)$ such that

$$\begin{pmatrix} g_{i,1}(z_i) = g_i(z_i) \\ \vdots \\ g_{i,m_i}(z_i)\end{pmatrix} = B_i(z_i)\begin{pmatrix} g_{i,1}\left(z_i^{q_i^{l_i}}\right) \\ \vdots \\ g_{i,m_i}\left(z_i^{q_i^{l_i}}\right)\end{pmatrix}, \tag{10.4.i}$$

$g_i(\alpha_i) = f_i(\alpha_i)$ and α_i is regular w.r.t. (10.4.i).

Let us first prove Case (i) of Theorem 1.1. Let us divide the natural numbers $1,\ldots,r$ into s classes $\mathcal{I}_1 = \{i_{1,1},\ldots,i_{1,\nu_1}\},\ldots,\mathcal{I}_s = \{i_{s,1},\ldots,i_{s,\nu_s}\}$, such that i and j belong to the same classe if and only if q_i and q_j are multiplicatively dependent. Iterating each system (10.4.i) a suitable number of times, we can assume without loss of generality that $q_i^{l_i} = q_j^{l_j} := \rho_k$ whenever i and j belong to $\mathcal{I}_k$. Set $\mathcal{E} := (g_1(\alpha_1),\ldots,g_r(\alpha_r))$ and
$$\mathcal{E}_k := (g_{i_{k,1}}(\alpha_{i_{k,1}}),\ldots,g_{i_{k,\nu_k}}(\alpha_{i_{k,\nu_k}}))\,,\ \forall k\in\{1,\ldots,s\}\,.$$
Given $k\in\{1,\ldots,s\}$, we consider the Mahler system in the variables $z_i, i\in\mathcal{I}_k$, associated with the matrix $\oplus_{i\in\mathcal{I}_k}B_i(z_i)$ and the transformation $T_k = \rho_k\mathrm{I}_{\nu_k}$, where we let I_n denote the identity matrix of size n. In this way, we have converted our r Mahler systems in one variable into s Mahler systems, each having respectively $\nu_1,\ldots,\nu_s$ variables. Furthermore, since by assumption the algebraic numbers $\alpha_1,\ldots,\alpha_r$ are multiplicatively independent, we deduce that each pair
$$(T_k, \boldsymbol{\alpha}_k := (\alpha_{i_{k,1}},\ldots,\alpha_{i_{k,\nu_k}}))\,,\ 1\le k\le s\,,$$

[8]Indeed, if $f(z)$ is a q-Mahler function then $f(z^{q^l})$ is clearly a q-Mahler function too.

is admissible. Finally, the point $\boldsymbol{\alpha}_k$ is regular for each α_i is regular w.r.t. (10.4.i). Since, by construction, the numbers $\rho(T_1) = \rho_1, \ldots, \rho(T_s) = \rho_s$ are pairwise multiplicatively independent, we can apply our second purity theorem, Theorem 2.8, to these s Mahler systems. We deduce that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{k=1}^{s} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k \mid \mathcal{E})\,. \tag{10.5}$$

Now, let us fix $k \in \{1, \ldots, s\}$. Since the numbers $\alpha_i, i \in \mathcal{I}_k$, are multiplicatively independent, we can apply our first purity theorem, Theorem 2.6, to the ν_k distinct Mahler systems (10.4.i), with $i \in \mathcal{I}_k$. For every $i \in \mathcal{I}_k$, set $\mathcal{E}_{k,i} := (g_i(\alpha_i))$. Since $g_i(\alpha_i) = f_i(\alpha_i)$ is transcendental, we have $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_{k,i}) = \{0\}$ for every $i \in \mathcal{I}_k$. We thus deduce from Theorem 2.6 that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k) = \sum_{i \in \mathcal{I}_k} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_{k,i} \mid \mathcal{E}_k) = \{0\}\,.$$

Since this holds for every k, $1 \leq k \leq s$, it follows from (10.5), that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \{0\}$. That is, $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$.

Now, let us prove Case (ii) of Theorem 1.1. As previously, we associate with each pair $(f_i(z), \alpha_i)$ a function $g_i(z)$ satisfying the conditions of Lemma 10.1. Since the natural numbers q_i are pairwise multiplicatively independent, we can apply our second purity theorem, Theorem 2.8, to the Mahler systems associated with each $g_i(z)$ in Lemma 10.1. Setting

$$\mathcal{E} := (g_1(\alpha_1), \ldots, g_r(\alpha_r))$$

and $\mathcal{E}_i := (g_i(\alpha_i))$, $1 \leq i \leq r$, we deduce that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})\,.$$

Again, since by assumption $g_i(\alpha_i) = f_i(\alpha_i)$ is transcendental, we get that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) = 0$ for every i, $1 \leq i \leq r$. This shows that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \{0\}$, and we conclude, as previously, that $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$. $\square$

Appendix A. Representing numbers in independent bases

In this appendix, we show how our main results apply to certain problems concerning expansions of numbers in multiplicatively independent bases. In particular, we state and prove Conjectures A.2, A.3, and Corollary A.4, which were our initial goal. All these results are deduced from Theorem 1.1.

A.1. **The dynamical point of view: Furstenberg's conjecture.** In the late 1960s, Furstenberg [26, 27] suggested a series of conjectures whose aim is to capture the heuristic which has been alluded to only in very vague terms at the beginning of this paper. These conjectures, which became famous, take place in a dynamical setting. This does not come as a great surprise for there is a well-known dictionary transferring combinatorial properties of the expansion of a real number x in an integer base q in terms of dynamical properties of the orbit of $\{x\}$ under the map T_q defined on $\mathbb{R}/\mathbb{Z}$ by $x \mapsto qx$. We let $\mathcal{O}_q(x)$ denote the forward orbit of x under T_q, that is,

$$\mathcal{O}_q(x) := \left\{x, T_q(x), T_q^2(x), \dots\right\} .$$

If $X \subset \mathbb{R}$, we let $\dim_H(X)$ denote the Hausdorff dimension of X and $\overline{X}$ its closure. One of Furstenberg's conjecture [27] reads as follows.

Conjecture A.1 (Furstenberg)**.** *Let p and q be two multiplicatively independent natural numbers, and let $x \in [0,1)$ be a real number. Then*

$$\dim_H \overline{\mathcal{O}_p(x)} + \dim_H \overline{\mathcal{O}_q(x)} \geq 1\,,$$

unless x is rational.

This conjecture has wonderful consequences about expansions of both real and natural numbers. It beautifully expresses the expected balance between the complexity of expansions of an irrational real number in two multiplicatively independent bases:

If x has a simple expansion in base p, then it should have a complex expansion in base q.

It is easy to see that Conjecture A.1 holds true generically. Indeed, endowed with the Haar measure, the topological dynamical system $(T_q, \mathbb{R}/\mathbb{Z})$ becomes ergodic, and it follows from the ergodic theorem that

$$\dim_H \overline{\mathcal{O}_p(x)} = \dim_H \overline{\mathcal{O}_q(x)} = 1\,,$$

for almost all real numbers x in $[0,1)$. In fact, all the strength of Conjecture A.1 takes shape when x has a simple expansion in one of the two bases. Defining the entropy of x with respect to the base q as the topological entropy of the dynamical system $(T_q, \overline{\mathcal{O}_q(x)})$, Conjecture A.1 implies that if x has zero entropy in base p, then it has a dense orbit under T_q.

Let us illustrate this with a concrete example. The binary Thue-Morse number τ is defined as follows. Its nth binary digit is equal to 0 if the sum of digits in the binary expansion of n is even, and to 1 otherwise. It is somewhat puzzling that its decimal expansion

$$\langle\tau\rangle_{10} = 0.412\,454\,033\,640\,107\,597\,783\,361\,368\,258\,455\,283\,089\cdots$$

seems unpredictable, while its binary expansion

$$\langle\tau\rangle_2 = 0.011\,010\,011\,001\,011\,010\,010\,110\,011\,010\,011\,001\,011\cdots$$

is, by definition, so regular. This intriguing phenomenon would be nicely explained by Conjecture A.1. Indeed, since τ has zero entropy in base 2, it should have a dense orbit under T_{10}, meaning that all blocks of digits should occur in its decimal expansion.

Other astonishing consequences of Conjecture A.1 concern expansions of natural numbers. For instance, using an elementary construction, Furstenberg shows in [27] how to deduce from Conjecture A.1 that any finite block of digits occurs in the decimal expansion of 2^n, as soon as n is large enough. Note that, in the same vein, a conjecture of Erdös claims that the digit 2 occurs in the ternary expansion of 2^n for all $n > 8$ (see, for instance, [32]).

Recently, Shmerkin [58] and Wu [61] proved that the set of exceptions to Conjecture A.1 has Hausdorff dimension zero. Unfortunately, this remarkable results does not tell us anything about expansions of real numbers with zero entropy in some base. Indeed, the set of all such real numbers has Hausdorff dimension zero [46]. Though the works of Shmerkin and Wu mark significant progress towards Conjecture A.1, the latter remains far out of the reach of current methods. Even worse, we are afraid that their result could be essentially the best dynamical methods have to say about this conjecture.

A.2. **The computational point of view: from finite automata to Mahler's method.** From a computational point of view, there is another relevant notion of simple number, namely the notion of *automatic real number* (see [12, Chap. 13]). While computable numbers can be generated by general Turing machines, automatic numbers are those whose expansion in some base can be generated by a finite automaton. Broadly speaking, a finite automaton is a Turing machine without any memory tape, all its memory being stored in the finite state control. This severe restriction justifies that these numbers are considered as especially simple. For example, the Thue-Morse number τ is automatic in base 2. We refer the reader to [5] and the references therein for a discussion on these different models of computation. In this new framework, our general heuristic naturally leads to the following conjecture.

Conjecture A.2. *Let p and q be two multiplicatively independent natural numbers. A real number cannot be automatic in both bases p and q, unless it is rational.*

This conjecture turns out to be a very special case of Conjecture A.1 for being automatic in some base implies having zero entropy in that base. Nevertheless, Conjecture A.2 remains quite challenging since, to date, not a single real number has been proved to be at once automatic in some base and not automatic in another one. We also mention that a weaker version of Conjecture A.2 appears as Open Problems 7 in [12, Chap. 13].

With a more Diophantine flavor, Conjecture A.2 can be strongly strengthened as follows.

Conjecture A.3. *Let $r \geq 1$ be an integer. Let $b_1, \ldots, b_r$ be multiplicatively independent positive integers, and, for every i, $1 \leq i \leq r$, let ξ_i be a real number that is automatic in base b_i. Then the numbers $\xi_1, \ldots, \xi_r$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them is rational.*

Conjecture A.3 is not implied by Furstenberg's conjecture. The former does not only imply that the Thue-Morse number τ cannot be automatic in base 10, but also that this is the case for any number obtained from τ by using algebraic numbers and algebraic operations (addition, multiplication, division, taking nth roots...). The case $r = 1$ was a long-standing conjecture first proved by Bugeaud and the first author [4] by means of the subspace theorem. See also [8, 53] for a recent different proof based on Mahler's method. So far, Conjecture A.3 has only be settled in that particular case.

A.2.1. *Connection with Mahler's method and Theorem 1.1.* In 1968, Cobham [20] first noticed the following fundamental connection between automatic sequences and M-functions. If the sequence $(a_n)_{n\geq 0}$ is q-automatic, then the generating function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

is a *q-Mahler function.* In turn, problems about transcendence and algebraic independence of automatic real numbers can be translated and extended to problems concerning transcendence and algebraic independence of M-functions at algebraic points. In particular, Conjecture A.3, and hence Conjecture A.2, easily follow from Part (i) of Theorem 1.1.

Proof of Conjecture A.3. Replacing $\xi_1, \ldots, \xi_r$ by their fractional part if necessary, we can assume without any loss of generality that $0 \leq \xi_i < 1$, for every i, $1 \leq i \leq r$. By assumption, the number ξ_i is automatic in base b_i. This means that, for some integer $q_i \geq 2$, there exists a q_i-automatic sequence $(a_{i,n})_{n\geq 0}$ with values in $\{0, 1, \ldots, b_i - 1\}$ such that $\xi_i = f_i(1/b_i)$, where $f_i(z) := \sum_{n=0}^{\infty} a_{i,n} z^n \in \mathbb{Q}\{z\}$. A discussed in [20], the fact that the sequence $(a_{i,n})_{n\geq 0}$ is q_i-automatic implies that $f_i(z)$ is a q_i-Mahler function. Now, let us assume that $\xi_1, \ldots, \xi_r$ are all irrational. By [8, Corollaire 1.8], we obtain that these numbers are all transcendental. Since by assumption the numbers $1/b_1, \ldots, 1/b_r$ are multiplicatively independent, Part (i) of Theorem 1.1 implies that the numbers $\xi_1 = f_1(1/b_1), \ldots, \xi_r = f_r(1/b_r)$ are algebraically independent, as wanted. $\square$

As with Furstenberg's conjecture, Theorem 1.1 has also valuable consequences about expansions of natural numbers. Let us first recall that a set $\mathcal{E} \subset \mathbb{N}$ is q-automatic if its elements, when written in base q, can be recognized by a finite automaton (See, for instance, [12, Chapter 5]). In this framework, there is a famous theorem by Cobham [21] that can be stated as follows. If $\mathcal{E} \subset \mathbb{N}$ is both p- and q-automatic, where p and q are multiplicatively independent, then $\mathcal{E}$ is a periodic set, meaning that $\mathcal{E}$ is the union of a finite set and finitely many arithmetic progressions.

Cobham's theorem can be rephrased in terms of power series. Indeed, it is equivalent to the fact that, given any aperiodic p-automatic set $\mathcal{E}_p$ and any aperiodic q-automatic set $\mathcal{E}_q$, the corresponding generating functions cannot be equal. That is,

$$\sum_{n\in\mathcal{E}_p} z^n =: f_p(z) \neq f_q(z) := \sum_{n\in\mathcal{E}_q} z^n .$$

In 1987, Loxton and van der Poorten [55] conjectured the following generalization: a power series in $\overline{\mathbb{Q}}[[z]]$ cannot be both p-Mahler and q-Mahler, unless it is rational. This conjecture was first proved by Bell and the first author in [2], while a different proof was given by Schäfke and Singer [57]. Very recently, the authors of [7] even proved a stronger result also conjectured by Loxton and van der Poorten [55]: a p-Mahler function $f_p(z) \in \overline{\mathbb{Q}}[[z]]$ and a q-Mahler function $f_q(z) \in \overline{\mathbb{Q}}[[z]]$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, unless one of them is rational. This result refines Cobham's theorem by expressing, in algebraic terms, the discrepancy between aperiodic automatic sets associated with multiplicatively independent input bases. The proof given in [7] is based on a suitable parametrized Galois theory associated with linear Mahler equations and follows the strategy initiated in [6].

Part (ii) of Theorem 1.1 leads to the following significant generalization of all the aforementioned results, providing in particular a totally new proof of Cobham's theorem.

Corollary A.4. *Let $r \geq 1$ be an integer. For every integer i, $1 \leq i \leq r$, let $q_i \geq 2$ be an integer and $f_i(z) \in \overline{\mathbb{Q}}\{z\}$ be a q_i-Mahler function. Assume that $q_1, \ldots, q_r$ are pairwise multiplicatively independent. Then $f_1(z), \ldots, f_r(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, unless one of them is rational.*

The case $r = 1$ is a classical result (see, for instance, [52, Theorem 5.1.7]), while, as previously mentioned, the case $r = 2$ is much harder and was only recently proved in [7].

Proof. Let us assume that the functions $f_1(z), \ldots, f_r(z)$ are all irrational. Then, by [52, Theorem 5.1.7], they are all transcendental over $\overline{\mathbb{Q}}(z)$. Combining Nishioka's theorem and [16, Lemma 6], we deduce that there exists $r > 0$ such that for all algebraic numbers α, with $0 < |\alpha| < r$, the numbers $f_1(\alpha), \ldots, f_r(\alpha)$ are all transcendental. Picking such α and applying Part (ii) of Theorem 1.1, we obtain that the numbers $f_1(\alpha), \ldots, f_r(\alpha)$ are algebraically independent over $\overline{\mathbb{Q}}$. Hence the functions $f_1(z), \ldots, f_r(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, as wanted. $\square$

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Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, F-69622 Villeurbanne Cedex, France

E-mail address: Boris.Adamczewski@math.cnrs.fr

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, F-69622 Villeurbanne Cedex, France

E-mail address: colin.faverjon@riseup.net