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# Reversals and palindromes in continued fractions

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**Abstract**

Several results on continued fractions expansions are on indirect consequences of the *mirror formula*. We survey occurrences of this formula for Sturmian real numbers, for (simultaneous) Diophantine approximation and for formal power series.


*Keywords:* Continued fractions; Combinatorics on words; Palindromes; Sturmian sequences; Mirror formula; Folding lemma; Diophantine approximation

## 1. Introduction

In the present survey, a conference version of which appeared as [1], we will focus on reversals of patterns and on palindromic patterns that occur in continued fraction expansions for real numbers and for formal Laurent series with coefficients in a finite field. Our main motivation comes from the remark that various very recent, and apparently unrelated, works make use of an elementary formula for continued fractions, referred to as the *mirror formula* all along this paper (see for example [3,4,6,5,7,15,19,21,22,44,75,74] for related papers published since 2005). This leads us to review some of these results, together with older ones, and to underline the central rôle played by this formula.

The first part of the paper (Sections 4 and 5) deals with combinatorics on words. We investigate in particular some questions related to the critical exponent, to the recurrence quotient, and to the palindrome density of sequences (also called infinite words). Most of the results involve Sturmian sequences: one characterization among others of these infinite words is that they are binary codings of non-periodic trajectories on a square billiard. The continued fraction expansion of the slope of these trajectories unveils the combinatorial properties of the associated Sturmian words, which explains that the mirror formula naturally appears in this framework.

The following sections are essentially devoted to *Diophantine approximation*, which can be defined as the art of answering the question: how good an approximation of a given real number by rationals $p/q$ as a function of $q$ can be? Continued fractions and Diophantine approximation are of course intimately connected, since the best rational approximations to a real number are produced by truncating its continued fraction expansion. It is

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however much less known, and quite new, that continued fractions can be used in order to study some questions of simultaneous approximation (i.e., the more general problem of approximating several real numbers by rationals having the same denominators). Mainly due to the lack of a suitable multi-dimensional continued fraction algorithm, such problems are generally considered as rather difficult. We will survey some old Diophantine questions together with recent developments where continued fractions, thanks to the mirror formula, are used to provide simultaneous rational approximations for some real numbers. In this regard, Section 6 is an exception since it deals with rational approximation of (only) one real number, defined by its binary expansion. However, Section 6 is still concerned by both Diophantine approximation and the mirror formula. Section 7 addresses simultaneous approximation for a number and its square. Section 8 deals with the Littlewood conjecture. Section 9 studies the transcendence of some families of continued fractions.

## 2. Notations

We will use the classical notations for finite or infinite continued fractions

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_0, a_1, \ldots, a_n]$$

resp.

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}} = [a_0, a_1, \ldots, a_n, \ldots]$$

where $p/q$ is a positive rational number, resp. $\alpha$ is a positive irrational real number, $n$ is a nonnegative integer, $a_0$ is a nonnegative integer, and the $a_i$'s are positive integers for $i \geq 1$. If $0 \leq k \leq n$, we denote by $p_k/q_k$ the $k$-th convergent to $p/q$ (resp. to $\alpha$), i.e., $p_k/q_k := [a_0, a_1, \ldots, a_k]$. In particular, for the rational $p/q$ we have $p/q = p_n/q_n = [a_0, a_1, \ldots, a_n]$. The sequence of denominators of the convergents to $p/q$ (resp. to $\alpha$) satisfies, for $n$ such that $1 \leq k \leq n$, the relation $q_k = a_k q_{k-1} + q_{k-2}$, with the convention that $q_{-1} := 0$ and $q_0 := 1$.

We will also have continued fractions for formal Laurent series over a field $K$: in this case, $p/q$ is a rational function ($p$ and $q$ are two polynomials in $K[X]$), resp. $\alpha$ is a Laurent series $\sum_{j \geq t} r_j X^{-j}$, $n$ is a nonnegative integer, and the $a_i$'s are nonzero polynomials in $K[X]$.

## 3. A fundamental lemma

A pleasant and useful formalism for continued fractions is the matrix formalism that we borrow from papers of van der Poorten (see, for example, [69,73]), who says that it goes back at least to [45]: we have that

$$\forall n \geq 0, \quad [a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}, \text{ with } \gcd(p_n, q_n) = 1$$

if and only if

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Taking the transposition of this equality easily yields the following lemma:

**Lemma 1** (*Mirror Formula*). *Let $a_0, a_1, \ldots$ be positive integers. Let $\frac{p_n}{q_n} := [a_0, a_1, \ldots, a_n]$. Then*

$$\frac{q_n}{q_{n-1}} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}} = [a_n, a_{n-1}, \ldots, a_1]. \tag{1}$$

We will call equality (1) the *mirror formula* throughout this paper. A useful variation on the mirror formula is known as the *folding lemma* (see [69,73] and Section 10). Another variation of the mirror formula is concerned with the *Lévy constant* of a continued fraction (see for example [14,39,78] for recent related works). Since this subject would deserve a whole survey, we restrict ourselves to point out the following relation:

$$\frac{1}{n}\log q_n = \frac{1}{n}\sum_{1\le k\le n}\log\frac{q_k}{q_{k-1}} = \frac{1}{n}\sum_{1\le k\le n}\log\,[a_k, a_{k-1}, \ldots, a_1]$$

(recall that the Lévy constant of a continued fraction with convergents $p_n/q_n$ is the limit, if it exists, of $q_n^{1/n}$, when $n$ tends to infinity).

## 4. Sturmian sequences and continued fractions

### 4.1. Generalities

Sturmian sequences can be defined in several ways. We choose the arithmetic definition. (For a general overview on Sturmian sequences, see for example [18].)

**Definition 2.** A sequence $(u_n)_{n\ge 0}$ is called *Sturmian* if there exist a positive irrational number $\alpha$ (called the *slope* of the Sturmian sequence) and a real number $\beta \in [0, 1)$ such that

- either $\forall n \ge 0$, $u_n = \lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor - \lfloor \alpha \rfloor$;
- or $\forall n \ge 0$, $u_n = \lceil \alpha(n+1) + \beta \rceil - \lceil \alpha n + \beta \rceil - \lceil \alpha \rceil$.

A sequence $(u_n)_{n\ge 0}$ is called *Sturmian characteristic* (or, simply, *characteristic*) if it is of the form above with $\beta = 0$.

**Remark 3.** Note that the definition shows that a Sturmian sequence takes its values in $\{0, 1\}$.

The following proposition shows a link between Sturmian sequences and continued fractions (see, for example, [11, Chapter 9] or [56, Chapter 2]). If $W$ is a finite word and $a$ a positive integer, we denote as usual by $W^a$ the concatenation of $a$ copies of the word $W$.

**Proposition 4.** *Let $\alpha = [0, a_1, a_2, \ldots]$ be an irrational number in $[0, 1)$. Define the sequence of words $(s_j)_{j\ge -1}$ by $s_{-1} := 1$, $s_0 := 0$, $s_1 := s_0^{a_1-1}s_{-1}$, and $s_j := s_{j-1}^{a_j}s_{j-2}$ for $j \ge 2$. Then the sequence $(s_j)_{j\ge 0}$ tends to an infinite word which is equal to the characteristic Sturmian word of slope $\alpha$.*

**Definition 5.** The Fibonacci sequence (or Fibonacci word) on the alphabet $\{0, 1\}$ is the characteristic Sturmian sequence defined as $\lim_{j\to+\infty} s_j$ where the words $s_j$ are defined by $s_{-1} := 1$, $s_0 := 0$, and, for all $j \ge 1$, $s_j := s_{j-1}s_{j-2}$. Hence this sequence begins as follows

0 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 . . . .

### 4.2. Repetitions in Sturmian sequences

Several authors have studied repetitions, i.e., factors (or sub-blocks) of the form $W^a$ occurring in a Sturmian sequence (see in particular the papers [63,64,16,24,95,23,33,19]). It happens that the mirror formula can be used in these studies. We give, as an example of repetitions in Sturmian words and the mirror formula, a (rephrasing of a) theorem due to Vandeth [95, Theorem 16]. First recall that the length of a (finite) word $W$ is denoted by $|W|$, and define *fractional powers* of finite words as follows: if $x$ is a positive real number and $W$ a finite word, then $W^x := W^{\lfloor x \rfloor}U$, where $U$ is the prefix of $W$ of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$. This notion goes back to Dejean [41] where she calls *sesquipuissances* what is now called fractional powers. We also define the *critical exponent* of an infinite word as the supremum of all powers occurring in this infinite word. This notion goes back to Mignosi and Pirillo [64].

**Theorem 6** (*Vandeth*). *Let $\alpha$ be the real number whose (eventually periodic) continued fraction expansion has the form*

$$\alpha = [0, b_0, b_1, b_2, \ldots, b_m, b_1, b_2, \ldots, b_m \ldots],$$

*where the $b_i$'s are positive integers and $b_m \geq b_0$ (in particular $\alpha$ is quadratic). Then the critical exponent of the characteristic Sturmian sequence $S_\alpha$ of slope $\alpha$ is*

$$\max_{1 \leq t \leq m} [2 + b_t, b_{t-1}, \ldots, b_1, b_m, \ldots, b_1, b_m, \ldots, b_1, \ldots]$$

**Remark 7.** – Note that the characteristic Sturmian sequences in Theorem 6 above are those that are fixed point of morphisms [31]. Vandeth deduces from his theorem the *integer* critical exponent of *any* characteristic Sturmian sequence $S_\alpha$ (even those that are not fixed points of morphisms) provided that $\alpha$ has bounded partial quotients (see [95, Theorem 17]). Also note that Carpi and de Luca give in [24] an expression for the critical exponent of any Sturmian sequence whose slope has ultimately periodic continued fraction expansion, i.e., is a quadratic surd.
– A fine study of the critical exponent (or *index*) and of the *initial critical exponent* of Sturmian sequences can be found in [19]. See also the second part of Remark 11.

### 4.3. Recurrence function, the Cassaigne spectrum

The *recurrence function* of an infinite sequence describes the size of maximal gaps between two occurrences of a same factor (sub-block) in the sequence. More formally

**Definition 8.** The recurrence function $R_u(n)$ of a sequence $u = (u_k)_{k \geq 0}$ is defined by: $R(n)$ is the smallest integer $m \leq +\infty$ such that each factor of length $m$ in the sequence $u$ contains all factors of length $n$ of $u$.

If $R_u(n) < +\infty$ for all $n$, the sequence is said to be *uniformly recurrent*. The *recurrence quotient* $\rho = \rho_u$ of the sequence $u$ is defined by $\rho_u := \limsup_{n \to +\infty} \frac{R_u(n)}{n}$.

**Remark 9.** It is clear that $\rho = +\infty$ if the sequence is not uniformly recurrent, and that $\rho = 1$ for a periodic sequence. If the sequence is not periodic it can be proven that $2 \leq \rho \leq +\infty$.

The following result, due to Cassaigne [28], makes use of the mirror formula:

**Theorem 10** (*Cassaigne*). *Let $u$ be a Sturmian sequence of slope $\alpha$. Let $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha$. Then*

$$\rho_u = 2 + \limsup_{i \to +\infty} [a_i, a_{i-1}, \ldots, a_1].$$

**Remark 11.** – Morse and Hedlund noted in [66] that $\rho_u = +\infty$ for almost all Sturmian sequences. The set $\{\rho_u,\ u \text{ Sturmian}\}$ is studied in more details in [28]. We will hence call it the *Cassaigne spectrum*. This set can be compared with, but is different from, the Lagrange and the Markoff spectra (see [32], for example).
– As noted in [19] (where the authors say this is a consequence of known results, for example [95]), *the limsup of powers of longer and longer words occurring in a Sturmian sequence of slope $\alpha = [a_0, a_1, \ldots]$ is equal to* $2 + \limsup[a_n, a_{n-1}, \ldots, a_1]$. An immediate nice consequence is that *the recurrence quotient of a Sturmian sequence and the limsup of powers of longer and longer words occurring in this sequence are equal*. As underlined by a referee this statement is an easy consequence of Propositions 8 and 11 of [24]. This leads to asking for which sequences this property does hold.
– A conjecture due to Rauzy asserts that the recurrence quotient of any nonperiodic infinite word is larger than $\frac{5+\sqrt{5}}{2}$ and that this value is optimal ([81] where the constant is misprinted as $\frac{3+\sqrt{5}}{2}$). The optimality of the constant comes

from the fact that it is the recurrence quotient of the Fibonacci sequence, fixed point of the morphism $0 \to 01$, $1 \to 0$. For related or similar questions, see [9,26,27].

## 5. Palindrome density

In this section, we consider palindromic prefixes of infinite words. Let us recall that a finite word $W = w_1 w_2 \cdots w_n$ is a palindrome if it is invariant under mirror symmetry, i.e., if it is equal to its *reversal*: $W = \overline{W}$, where $\overline{W} := a_n a_{n-1} \cdots a_1$. Let $w = w_1 w_2 \cdots w_n \cdots$ be an infinite word beginning in arbitrarily long palindromes. For such a word, let us denote by $(n_i)_{i \geq 1}$ the increasing sequence of all lengths of palindromic prefixes of $w$. By assumption, this sequence is thus infinite. In [44], Fischler defines the *palindrome density* of $w$, denoted $d_p(w)$, by

$$d_p(w) := \left( \limsup_{i \to \infty} \frac{n_{i+1}}{n_i} \right)^{-1}$$

(where $d_p(w) := 0$ if the word $w$ begins in only finitely many palindromes). Clearly $0 \leq d_p(w) \leq 1$. Furthermore, if $w = ZZZ \cdots$ is a periodic word, then $d_p(w) = 1$ if there exist two (possibly empty) palindromes $U$ and $V$ such that $Z = UV$, and $d_p(w) = 0$ otherwise. Thus the palindrome density of periodic infinite words is either maximal or minimal. This naturally leads to the following question: what is the maximal palindrome density that can be attained by an non-periodic infinite word? This problem is solved in [44].

**Theorem 12** (*Fischler*). *Let $w$ be an infinite non-periodic word. Then,*

$$d_p(w) \leq \frac{1}{\gamma},$$

*where $\gamma := \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

The bound obtained in Theorem 12 is optimal and reached in particular for the Fibonacci word. More generally, it is possible to compute $d_p(w)$ when $w$ is a characteristic Sturmian word. Indeed, if $\alpha = [0, a_1, a_2, \ldots]$ denotes a real number and if $w_\alpha$ is the associated characteristic Sturmian sequence, then

$$d_p(w_\alpha) = \frac{\sigma + 1}{2\sigma + 1},$$

where $\sigma := \limsup_{n \to \infty} [a_n, a_{n-1}, \ldots, a_1]$. In other words, the computation of the palindrome density of a characteristic Sturmian sequence involves the mirror formula, *via* the convergents to its slope.

**Remark 13.** – As suggested by a referee of the conference version of this paper, it is interesting to observe that the characteristic Sturmian words whose continued fraction expansion of the slope begins in arbitrarily long palindromes are exactly the standard infinite harmonic words introduced in [25].
– It might be of interest to note that the Fibonacci sequence again (see third part of Remark 11) enters the picture as an "extremal word".

We now introduce a modification of the Cassaigne spectrum (see Remark 11). Let $\mathcal{S}'_c$ be defined by

$$\mathcal{S}'_c := \left\{ d_p(w_\alpha),\ \alpha \in (0, 1) \setminus \mathbb{Q} \right\} = \left\{ \frac{\sigma + 1}{2\sigma + 1},\ \sigma \in \mathcal{S}_c \right\}.$$

We also denote by $\mathcal{S}_p$ the set of the real numbers that can be written $d_p(w)$ for some infinite word $w$. The following interesting result is proved in [44]: if a non-periodic word $w$ has a palindrome density that is "too large", then there exists an irrational number $\alpha$ such that $d_p(w) = d_p(w_\alpha)$. This can be formalized as follows:

**Theorem 14** (*Fischler*). *We have*

$$\mathcal{S}_p \cap \left[ \frac{1}{\sqrt{3}}, 1 \right) = \mathcal{S}'_c \cap \left[ \frac{1}{\sqrt{3}}, 1 \right).$$

**Remark 15.** A similar question arises in relation with Rauzy's conjecture recalled in Remark 11: is it true that any recurrence quotient sufficiently close to $\frac{5+\sqrt{5}}{2}$ is actually the recurrence quotient of some Sturmian sequence, i.e., belongs to the Cassaigne spectrum?

We end this section by mentioning that the motivation for studying the palindrome density of infinite words comes from a problem of uniform simultaneous rational approximation, see Section 7.

## 6. Exact irrationality measure

The *irrationality measure* of an irrational real number $\alpha$, denoted by $\mu(\alpha)$, is defined as the supremum of the positive real numbers $\tau$ for which the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\tau}$$

has infinitely many solutions $(p, q) \in \mathbb{Z}^2$. Thus, $\mu(\alpha)$ measures the quality of the best rational approximations to $\alpha$. The theory of continued fractions ensures that $\mu(\alpha) \geq 2$, for any irrational number $\alpha$. Algebraic irrational numbers have irrationality measure 2, as follows from Roth's theorem [82]. This is also the case for almost all real numbers with respect to the Lebesgue measure (Khintchine [47], see also [49]). Let us also mention that Liouville numbers are defined as the real numbers having an infinite irrationality measure.

It is in general a challenging problem to compute or even to bound the irrationality measure of a given real number. In this section we consider a particular class of irrational numbers having the spectacular property that both their $b$-adic expansion and their continued fraction expansion can be explicitly determined. We will deduce from this last representation the exact value of their irrationality measure.

With an irrational number $\alpha$ and an integer $b$, both larger than 1, we associate the real number $S_b(\alpha)$ defined by

$$S_b(\alpha) := (b-1)\sum_{n=1}^{+\infty} \frac{1}{b^{\lfloor n\alpha \rfloor}}.$$

The following nice result can be found in [8] (see also [37]).

**Theorem 16** (*Adams & Davison*). *Let $\alpha := [a_0, a_1, a_2, \ldots]$ be a positive irrational number and b be an integer, both larger than* 1. *Let $p_n/q_n$ be the n-th convergent to $1/\alpha$. For $n \geq 1$, set*

$$t_n := (b^{q_n} - b^{q_{n-2}})/(b^{q_{n-1}} - 1).$$

*Then,*

$$S_b(\alpha) = [0, t_1, t_2, \ldots, t_n, \ldots].$$

We easily deduce from Theorem 16 and the mirror formula, the exact irrationality measure for $S_b(\alpha)$ for any irrational $\alpha$ and any integer $b$, both larger than 1.

**Theorem 17.** *Let $\alpha := [a_0, a_1, a_2, \ldots]$ be a positive irrational number and b be an integer, both larger than* 1. *Then*

$$\mu(S_b(\alpha)) = 1 + \limsup_{n\to\infty}\ [a_n, a_{n-1}, \ldots, a_0].$$

Let us remark that, up to a translation, the set $\mathcal{M} := \{\mu(S_b(\alpha)), \alpha \notin \mathbb{Q}\}$ is equal to the Cassaigne spectrum (see Remark 11). As a consequence, we always have that $\mu(S_b(\alpha)) \geq \frac{3+\sqrt{5}}{2} > 2$. In virtue of Roth's theorem, $S_b(\alpha)$ is thus transcendental.

**Proof of Theorem 17.** We keep the notations of Theorem 16. For any nonnegative integer, let us denote by $P_n/Q_n$ the $n$-th convergent to $S_b(\alpha)$. By Theorem 16, we know that $(Q_n)_{n\geq 0}$ is the sequence defined by

$$Q_0 := 1,\ Q_1 := 1, \ \text{ and for } n \geq 2, \quad Q_{n+1} := t_{n+1}Q_n + Q_{n-1}.$$

We first observe that, for any nonnegative integer $n$, $Q_n = (b^{q_n} - 1)/(b-1)$. Namely, for $n = 0$ and $n = 1$, this follows from $q_0 = 0$ and $q_1 = 1$. For $n \geq 2$, this is implied by

$$t_{n+1}\left(\frac{b^{q_n}-1}{b-1}\right)+\frac{b^{q_{n-1}}-1}{b-1} = \left(\frac{b^{q_{n+1}}-b^{q_{n-1}}}{b^{q_n}-1}\right)\left(\frac{b^{q_n}-1}{b-1}\right)+\frac{b^{q_{n-1}}-1}{b-1}$$

$$= \frac{b^{q_{n+1}}-b^{q_{n-1}}}{b-1}+\frac{b^{q_{n-1}}-1}{b-1}=\frac{b^{q_{n+1}}-1}{b-1}\cdot$$

On the other hand, the theory of continued fractions gives

$$\frac{1}{2Q_nQ_{n+1}} < \left|S_b(\alpha)-\frac{P_n}{Q_n}\right| < \frac{1}{Q_nQ_{n+1}}\cdot \tag{2}$$

This can be expressed as follows:

$$\frac{1}{Q_n^{1+(\log Q_{n+1}/\log Q_n)+(\log 2/\log Q_n)}} < \left|S_b(\alpha)-\frac{P_n}{Q_n}\right| < \frac{1}{Q_n^{1+(\log Q_{n+1}/\log Q_n)}}\cdot$$

Furthermore, we have that $\log Q_n = \log(b^{q_n}-1) - \log(b-1)$, which implies

$$\limsup_{n\to\infty}\frac{\log Q_{n+1}}{\log Q_n} = \limsup_{n\to\infty}\frac{q_{n+1}}{q_n}\cdot$$

We thus can precisely estimate the quality of approximations of $S_b(\alpha)$ by the rationals $P_n/Q_n$. From (2) and the mirror formula, we deduce that the inequality

$$\left|S_b(\alpha)-\frac{P_n}{Q_n}\right| < \frac{1}{Q_n^{\tau}}$$

has infinitely many solutions as soon as

$$\tau < 1+\limsup_{n\to\infty}[a_n, a_{n-1}, \ldots, a_0],$$

whereas it has only finitely many solutions if

$$\tau > 1+\limsup_{n\to\infty}[a_n, a_{n-1}, \ldots, a_0].$$

Since the rationals $P_n/Q_n$ are by definition the best rational approximations to $S_b(\alpha)$, we get that $\mu(S_b(\alpha)) = 1+\limsup_{n\to\infty}[a_n, a_{n-1}, \ldots, a_0]$, concluding the proof. □

**Remark 18.** For other results on Diophantine approximation of Sturmian numbers, see [79] and its bibliography.

## 7. Simultaneous approximation for a number and its square

The study of approximations to a real number by algebraic numbers of bounded degree began with Wirsing's 1960 paper [96]. He proved that if $n$ is an integer at least equal to 2, and if $\xi$ is not an algebraic number of degree at most $n$, there are infinitely many algebraic numbers $\alpha$ of degree at most $n$ satisfying

$$|\xi-\alpha| \ll H(\alpha)^{-(n+3)/2} \tag{3}$$

where $H(\alpha)$ denotes the *height* of $\alpha$, i.e., the largest absolute value of the coefficients of its irreducible polynomial over $\mathbb{Z}$. The constant implied by the notation $\ll$ depends on $n$ and $\xi$. A famous conjecture, due to Wirsing [96], claims that the right exponent in (3) is equal to $n+1$ instead of $(n+3)/2$. Up to now, the Wirsing conjecture is only known to be true for $n = 2$; this is a result of Davenport and Schmidt [35].

In 1969, Davenport and Schmidt [36] investigated the same question but with algebraic numbers replaced by algebraic integers. In the rest of this section, we will focus on the approximation to a real number by cubic integers, i.e., on a question related to the case $n = 3$ in (3). In this direction, Davenport and Schmidt [36] proved the following result:

**Theorem 19** (*Davenport and Schmidt*). *Let $\gamma := \frac{1+\sqrt{5}}{2}$. Let $\xi$ be a real number that is neither rational nor quadratic. Then, there exist a positive constant $c_1$ and infinitely many algebraic integers $\alpha$ of degree at most 3 such that*

$$|\xi - \alpha| \le c_1 H(\alpha)^{-\gamma^2},$$

*where $H(\alpha)$ denotes the height of the algebraic number $\alpha$.*

By a kind of "duality", approximation to a real number by algebraic numbers of bounded degree is also intimately connected with simultaneous uniform rational approximation to successive powers of a real number. In particular, approximation to a real number $\xi$ by algebraic cubic integers is related to simultaneous uniform rational approximation to $\xi$ and $\xi^2$, and the authors of [36] actually derive Theorem 19 from the following result.

**Theorem 20** (*Davenport and Schmidt*). *Let $\gamma := \frac{1+\sqrt{5}}{2}$. Let $\xi$ be a real number that is neither rational nor quadratic. Then, there exist a positive constant $c_2$ and arbitrarily large values of $X$ such that the inequalities*

$$|x_0| \le X,\ |x_0\xi - x_1| \le c_2 X^{-1/\gamma},\ |x_0\xi^2 - x_2| \le c_2 X^{-1/\gamma},$$

*do not have any nonzero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$.*

Until recently it was believed that the value $\gamma^2$ in Theorem 19 could be improved to 3. This is, however, not true, as discovered by Roy [84]. Actually, Roy proves the surprising result that the value $\gamma^2$ in Theorem 19 is optimal.

**Theorem 21** (*Roy*). *Let $\gamma := \frac{1+\sqrt{5}}{2}$. Then there exist a positive constant $c_3$ and a real number $\xi$ that is neither rational nor quadratic, such that for any algebraic integer $\alpha$ of degree at most 3, we have*

$$|\xi - \alpha| \ge c_3 H(\alpha)^{-\gamma^2}.$$

To obtain this result, Roy [83,85] first proves that the value $\gamma$ is in fact optimal in Theorem 20, against the natural conjecture that the value $\gamma$ could be improved to 2.

**Theorem 22** (*Roy*). *Let $\gamma := \frac{1+\sqrt{5}}{2}$. Then there exist a positive constant $c_4$ and a real number $\xi$ that is neither rational nor quadratic, such that the inequalities*

$$|x_0| \le X,\ |x_0\xi - x_1| \le c_4 X^{-1/\gamma},\ |x_0\xi^2 - x_2| \le c_4 X^{-1/\gamma},$$

*have a nonzero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for any real number $X > 1$.*

Following Roy, a real number satisfying the exceptional Diophantine conditions of Theorem 22 is called an *extremal number*. It is proved in [85] that the set of extremal numbers is countable. Surprisingly, Roy provides the following "natural" example of an extremal real number. Let $a$ and $b$ be two distinct positive integers. Let

$$\xi := [a, b, a, a, b, a, b, a, a, b, \ldots],$$

where $abaababaab\ldots$ denotes the Fibonacci word over the alphabet $\{a, b\}$ (see Definition 5 in Section 4). Then, Roy proved [85] that $\xi$ is an extremal number.

Of course the attractive work of Roy led to many stimulating questions. For a real number $\xi$ we define, following [21], the exponent $\hat{\lambda}_2(\xi)$ as the supremum of the real numbers $\lambda$ such that the inequalities

$$|x_0| \le X,\ |x_0\xi - x_1| \le c_4 X^{-1/\lambda},\ |x_0\xi^2 - x_2| \le c_4 X^{-1/\lambda},$$

have a nonzero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for any large enough real number $X$. Bugeaud and Laurent [21] showed how to use Roy's construction to provide explicit real numbers for which the exponent $\hat{\lambda}_2$ takes values beetwen 2 (the expected value) and $\gamma$ (the optimal value).

**Theorem 23** (*Bugeaud and Laurent*). *Let $m$ and $n$ be two distinct positive integers. Let $\alpha := [0, a_1, a_2, \ldots]$ be an irrational real number and let $(b_n)_{n\ge 1}$ be the characteristic Sturmian sequence of slope $\alpha$ defined on the alphabet $\{m, n\}$. Let $\xi$ be the non-quadratic and irrational real number defined by*

$$\xi := [0, b_1, b_2, \ldots].$$

*Then,*

$$\hat{\lambda}_2(\xi) = \frac{\sigma + 1}{2\sigma + 1},$$

*where* $\sigma := \limsup_{n\to\infty} [a_n, a_{n-1}, \ldots, a_1]$.

We end this section with a focus on the main steps of the proof of Theorem 22. Our presentation is quite far from the proof in [85] but quite close to that of [83], it shows how the mirror formula can play here a central rôle.

**Proof of Theorem 22.** We first need a lemma whose proof can be found for example in the book of Perron [67].

**Lemma 24.** *For any positive integers* $a_1, \ldots, a_m$ *and any integer* $k$ *with* $1 \le k \le m-1$*, denote by* $K_m(a_1, \ldots, a_m)$ *their* continuant*, i.e., the denominator of the rational number* $[0, a_1, \ldots, a_m]$*. We have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

*and*

$$1 \le \frac{K_m(a_1, \ldots, a_m)}{K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m)} \le 2.$$

Now, the proof of Theorem 22 can essentially be divided into three steps.

In the first and more important step, we show how continued fractions can be used for finding simultaneous rational approximations to a real number and its square, *via* palindromes. Let $\xi = [0, a_1, a_2, \ldots]$ be a positive irrational real number, and denote by $p_n/q_n$ its convergents, i.e., $p_n/q_n := [0, a_1, \ldots, a_n]$. If the word $a_1 \cdots a_n$ is a palindrome, then the mirror formula implies that

$$\frac{q_{n-1}}{q_n} = [0, a_n, a_{n-1}, \ldots, a_1] = [0, a_1, \ldots, a_n] = \frac{p_n}{q_n}\cdot$$

In this case, we have $p_n = q_{n-1}$. By the theory of continued fractions, we get

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \quad \text{and} \quad \left|\xi - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_{n-1}^2}\cdot$$

We then infer from $0 < \xi < 1$, $a_1 = a_n$ and $q_n \le (a_n + 1)q_{n-1}$ that

$$\begin{aligned}\left|\xi^2 - \frac{p_{n-1}}{q_n}\right| &\le \left|\xi^2 - \frac{p_{n-1}}{q_{n-1}} \times \frac{p_n}{q_n}\right| \le \left|\xi + \frac{p_{n-1}}{q_{n-1}}\right| \times \left|\xi - \frac{p_n}{q_n}\right| + \frac{1}{q_n q_{n-1}}\\ &\le 2\left|\xi - \frac{p_n}{q_n}\right| + \frac{1}{q_n q_{n-1}} < \frac{a_1 + 3}{q_n^2}\cdot\end{aligned}$$

Consequently, if the word $a_1 a_2 \cdots a_n$ is a palindrome, then

$$|q_n\xi - p_n| < \frac{1}{q_n} \quad \text{and} \quad |q_n\xi^2 - p_{n-1}| < \frac{a_1 + 3}{q_n}\cdot \tag{4}$$

In other words, each time a convergent $p_n/q_n$ to the real $\xi$ is palindromic (i.e., $p_n/q_n = [0, a_1, \ldots, a_n]$ and $a_1 \cdots a_n$ is a palindrome), it provides very good simultaneous rational approximations to $\xi$ and $\xi^2$, respectively given by $p_n/q_n$ and $p_{n-1}/q_n$.

An important feature of the problem we are studying is that we have to prove a uniform statement, that is, it deals with uniform simultaneous rational approximation. In the second step, which will now appear as very natural, we show how the palindrome density of the continued fraction expansion of a real $\xi$ is related to such a uniform statement.

First, let us assume that the infinite word $a = a_1 a_2 \cdots a_n \cdots$ begins in infinitely many palindromes. We will use the notation introduced in Section 5. We thus denote by $(n_i)_{i\ge1}$ the increasing sequence of all lengths of palindromic prefixes of $a$, and by $d_p(a)$ the palindrome density of the word $a$. Let us assume that the palindrome density of $a$

is large enough to ensure that $q_{n_{i+1}} \leq c q_{n_i}^{\tau}$, for some real number $\tau$ larger than one and for a positive constant $c$ independent of $i$. Then, it easily follows from (4) that for any real number $X > 1$, the inequalities

$$|x_0| \leq X, \ |x_0\xi - x_1| \leq cX^{-1/\tau}, \ |x_0\xi^2 - x_2| \leq cX^{-1/\tau}, \tag{5}$$

have a nonzero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. Indeed, given $X$ there always exists a positive integer $n$ such that $q_n \leq X < q_{n+1}$, and the triple $(q_n, p_n, p_{n-1})$ is a nonzero solution for (5).

Thus, if $\xi$ is a real number whose continued fraction expansion begins in many palindromes, then $\xi$ and $\xi^2$ are uniformly and simultaneously well approximated by rationals. In view of Section 5, the Fibonacci continued fraction thus appears as a natural candidate for our problem. This ends our second step.

From now on, we assume that $a = abaab\ldots$ denotes the Fibonacci word over the alphabet $\{a, b\}$ and that $\xi := [a, b, a, a, b, \ldots]$. We want to prove that $\xi$ is an extremal number. We thus have first to estimate the growth of the sequence $(n_i)_{i\geq 1}$. Actually as noted in [44] the value of $n_i$ can be computed exactly as a consequence of [57, Theorem 5]: we have

$$n_i = F_{i+1} - 2, \tag{6}$$

where $F_i$ denotes the $i$-th Fibonacci number.

To end the proof, it now suffices (in view of (5)) to prove that there exists a positive constant $c$ independent of $i$ such that

$$q_{n_{i+1}} \leq c q_{n_i}^{\gamma}, \tag{7}$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ denotes as previously the golden ratio. Lemma 24 and equality (6) imply that

$$c_5 < \frac{q_{n_{i+1}}}{q_{n_i} q_{n_{i-1}}} < c_6,$$

for any $i \geq 2$ and for some positive constants $c_5$ and $c_6$. We set

$$c_7 = \max\left\{c_6, \frac{(c_5 q_{n_1})^{\gamma}}{q_{n_2}}, \frac{(c_5 q_{n_2})^{1/\gamma}}{q_{n_1}}\right\}.$$

Since $c_7 \geq c_6$, we obviously get that

$$c_5 < \frac{q_{n_{i+1}}}{q_{n_i} q_{n_{i-1}}} < c_7. \tag{8}$$

We set $c_8 = c_5^{\gamma}/c_7$ and $c_9 = c_7^{\gamma}/c_5$, and we are now going to prove by induction on $i$ that

$$c_8 q_{n_i}^{\gamma} \leq q_{n_{i+1}} \leq c_9 q_{n_i}^{\gamma} \tag{9}$$

holds for any $i \geq 2$. For $i = 2$, this follows from (8) and from the definition of $c_7$. Let us assume that (9) holds for a fixed integer $i \geq 2$. By (8), we have

$$c_5 q_{n_i}^{\gamma}\left(q_{n_i}^{1-\gamma} q_{n_{i-1}}\right) < q_{n_{i+1}} < c_7 q_{n_i}^{\gamma}\left(q_{n_i}^{1-\gamma} q_{n_{i-1}}\right)$$

and since $\gamma(\gamma - 1) = 1$, we obtain

$$c_5 q_{n_i}^{\gamma}\left(q_{n_i} q_{n_{i-1}}^{-\gamma}\right)^{1-\gamma} < q_{n_{i+1}} < c_7 q_{n_i}^{\gamma}\left(q_{n_i} q_{n_{i-1}}^{-\gamma}\right)^{1-\gamma}.$$

We thus deduce from (9) that

$$\left(c_5 c_9^{1-\gamma}\right) q_{n_i}^{\gamma} < q_{n_{i+1}} < \left(c_8 c_5^{1-\gamma}\right) q_{n_i}^{\gamma}.$$

By definition of $c_8$ and $c_9$, and since $\gamma(\gamma - 1) = 1$, this gives

$$c_8 q_{n_i}^{\gamma} < q_{n_{i+1}} < c_9 q_{n_i}^{\gamma}.$$

We thus have shown that (9) holds for any integer $i \geq 2$. In virtue of (7) and (5), $\xi$ is an extremal number, which concludes the proof of Theorem 22. □

## 8. The Littlewood conjecture

It follows from the theory of continued fractions that, for any real number $\alpha$, there exist infinitely many positive integers $q$ such that

$$q \cdot \|q\alpha\| < 1, \tag{10}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. In particular, for any given pair $(\alpha, \beta)$ of real numbers, there exist infinitely many positive integers $q$ such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| < 1.$$

A famous open problem in simultaneous Diophantine approximation, called the Littlewood conjecture (see for example [55]), claims that in fact, for any given pair $(\alpha, \beta)$ of real numbers, a stronger result holds.

**Littlewood's conjecture.** *For any given pair $(\alpha, \beta)$ of real numbers,*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0. \tag{11}$$

Let us denote by Bad the set of badly approximable numbers, i.e.,

$$\mathrm{Bad} := \{\alpha \in \mathbb{R} : \inf_{q\geq 1} q \cdot \|q\alpha\| > 0\}.$$

The set Bad is intimately connected with the theory of continued fractions. Indeed, a real number lies in Bad if and only if it has bounded partial quotients in its continued fraction expansion. It then follows that the Littlewood conjecture holds true for the pair $(\alpha, \beta)$ if $\alpha$ or $\beta$ has unbounded partial quotients in its continued fraction expansion. It also holds when the numbers $1, \alpha$, and $\beta$ are linearly dependent over the rational integers, as follows from (10).

The first significant contribution towards the Littlewood conjecture goes back to Cassels and Swinnerton-Dyer [29] who showed that (11) holds when $\alpha$ and $\beta$ belong to the same cubic field. However, since it is still not known whether cubic real numbers have bounded partial quotients or not (see the discussion at the beginning of Section 9), their result does not yield examples of pairs of badly approximable real numbers for which the Littlewood conjecture holds.

In view of the above discussion, it is natural to restrict our attention to independent parameters $\alpha$ and $\beta$, both lying in Bad. This naturally leads to considering the following problem:

**Question 25.** *Given $\alpha$ in* Bad*, is there any independent $\beta$ in* Bad *so that the Littlewood conjecture is true for the pair $(\alpha, \beta)$?*

Apparently, Question 25 remained unsolved until 2000, when Pollington and Velani [68] gave a positive answer, by establishing the following stronger result:

**Theorem 26** (*Pollington and Velani*). *Given $\alpha$ in* Bad*, there exists a subset $A(\alpha)$ of* Bad *with Hausdorff dimension one, such that, for any $\beta$ in $A(\alpha)$, there exist infinitely many positive integers $q$ with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{\log q}. \tag{12}$$

*In particular, the Littlewood conjecture holds for the pair $(\alpha, \beta)$ for any $\beta$ in $A(\alpha)$.*

The proof of this result depends on sophisticated tools from metric number theory. At the end of [68], Pollington and Velani give an alternative proof of a weaker version of Theorem 26, namely with (12) replaced by (11). However, even for establishing this weaker version, deep tools from metric number theory are still needed, including in particular a result of Davenport, Erdős and LeVeque on uniform distribution [34] and the *Kaufman measure* constructed in [46].

Very recently, Einsiedler, Katok and Lindenstrauss [43] proved the following remarkable result:

**Theorem 27** (*Einsiedler, Katok and Lindenstrauss*). *The set of pairs of real numbers for which the Littlewood conjecture does not hold has Hausdorff dimension zero.*

Obviously, this gives a positive answer to Question 25. Actually, the authors established part of the Margulis conjecture on ergodic actions on the homogeneous space $SL_k(\mathbb{R})/SL_k(\mathbb{Z})$, for $k \geq 3$ (see [59]). It was previously known that such a result would have implications to Diophantine questions, including the Littlewood conjecture. Their sophisticated proof used, among others, deep tools from algebra and from the theory of dynamical systems, involving in particular the important work of Ratner (see for example [80]).

De Mathan gave in [60] an explicit construction of pairs of real numbers $(\alpha, \beta)$ with bounded partial quotients, such that $1, \alpha, \beta$ are linearly independent over the rationals and satisfy Littlewood's conjecture. We do not resist to give the following natural example derived by de Mathan in [60, p. 264] from Queffélec's paper [77]. Recall that the Thue–Morse sequence on the alphabet $\{1, 2\}$ is the sequence $(a_n)_{n\geq 0}$ defined by $a_n := 1$ (resp. $a_n := 2$) if the sum of the binary digits of $n$ is even (resp. odd).

**Theorem 28** (*de Mathan*). *Let $\eta := [1, 2, 2, 1, 2, 1, 1, 2, 2, \ldots]$ be the real number whose continued fraction expansion is the Thue–Morse sequence on the alphabet $\{1, 2\}$. Then, $1, \eta, 1/\eta$ are linearly independent on the rationals, and $(\eta, 1/\eta)$ satisfies the Littlewood conjecture.*

Actually our favourite formula was used by the authors of [3] to provide a short and elementary positive answer to Question 25, and even a stronger form of it. Their approach, based on the basic theory of continued fractions, gives a generic way to provide explicit examples for the Littlewood conjecture.

**Theorem 29** (*Adamczewski and Bugeaud*). *Let $\varphi$ be a positive and non-increasing function defined on the set of positive integers such that $\varphi(1) = 1$, $\lim_{q\to+\infty} \varphi(q) = 0$ and $\lim_{q\to+\infty} q\varphi(q) = +\infty$. Given $\alpha$ in* Bad*, there exists an uncountable subset $B_\varphi(\alpha)$ of* Bad *such that, for any $\beta$ in $B_\varphi(\alpha)$, there exist infinitely many positive integers $q$ with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{q \cdot \varphi(q)}. \tag{13}$$

*In particular, the Littlewood conjecture holds for the pair $(\alpha, \beta)$ for any $\beta$ in $B_\varphi(\alpha)$. Furthermore, the set $B_\varphi(\alpha)$ can be effectively constructed.*

It is of interest to compare this result with Theorem 26. Regarding the Littlewood conjecture, Theorem 26 is stronger since the set $A(\alpha)$ has Hausdorff dimension one whereas the set $B_\varphi(\alpha)$ has only the power of the continuum. On the other hand, one can remark that the Diophantine property in Theorem 29 is really stronger than the one of Theorem 26. In particular, one can doubt on the truth of a statement analogous to Theorem 26, with the Diophantine condition of Theorem 29.

## 9. Transcendental continued fractions

It is widely believed (the question was first asked by Khintchine [48] cited in [90]) that the continued fraction expansion of any irrational algebraic number $\alpha$ is either eventually periodic (and this is the case if and only if $\alpha$ is a quadratic irrational) or it contains arbitrarily large partial quotients; but we seem to be very far away from a proof (or a disproof). A first step consists in providing explicit examples of transcendental continued fractions. The first result of this type is due to Liouville [54], who constructed real numbers whose sequence of partial quotients grows very fast, too fast for the numbers to be algebraic. Subsequently, various authors used deeper transcendence criteria from Diophantine approximation to construct other classes of transcendental continued fractions. Of particular interest is the work of Maillet [58] (see also Section 34 of Perron [67]), who was the first to give examples of transcendental continued fractions with bounded partial quotients. Further examples were provided by Baker [12,13], Shallit [89], Davison [38], Queffélec [77], Allouche, Davison, Queffélec and Zamboni [10] and Adamczewski and Bugeaud [2], among others. Note that the folding lemma is used by Shallit in [89] (see also Section 10).

In the previous two sections, we have shown how the mirror formula can be used to find simultaneous rational approximations for some real numbers. On the other hand, algebraic numbers cannot be "too well" simultaneously approximated by rationals. This is a multi-dimensional Roth's principle (see Theorem 30 below). Such considerations give naturally rise to transcendence statements, as we will see in this section. We will first be interested in a familly of "quasi-periodic" continued fractions introduced by Maillet [58] and studied later by Baker in [12] and [13]. Then, we will investigate real numbers whose sequence of partial quotients enjoys another combinatorial property, namely is "symmetrical", in the sense that it begins in arbitrarily long palindromes or quasi-palindromes.

The transcendence criteria presented in this section rest on the powerful Subspace Schmidt Theorem [87] (see also [88]) that we state now, as well as on a heavy use of the mirror formula.

**Theorem 30** (*W.M. Schmidt*). *Let $m \geq 2$ be an integer. Let $L_1, \ldots, L_m$ be linearly independent linear forms in $x = (x_1, \ldots, x_m)$ with algebraic coefficients. Let $\varepsilon$ be a positive real number. Then, the solutions $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$ to the inequality*

$$|L_1(x) \cdots L_m(x)| \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

*lies in the union of finitely many proper subspaces of $\mathbb{Q}^m$.*

As an example of a by-product of the Subspace Theorem, we mention a result concerning the simultaneous rational approximation of a real number and its square. It was originally proved in [86].

**Theorem 31** (*W.M. Schmidt*). *Let $\xi$ be a real number, which is neither rational, nor quadratic. If there exist a real number $w > 3/2$ and infinitely many triples of integers $(p, q, r)$ such that*

$$\max\left\{\left|\xi - \frac{p}{q}\right|, \left|\xi^2 - \frac{r}{q}\right|\right\} < \frac{1}{|q|^w},$$

*then $\xi$ is transcendental.*

A direct consequence of this result is that the extremal numbers considered in Section 7 are transcendental. The following dual form of Theorem 31, also proved in [86], which limits the approximation of an algebraic nonquadratic number by quadratic numbers, can also be derived from the Suspace Theorem.

**Theorem 32** (*W.M. Schmidt*). *Let $\xi$ be a real number, which is neither rational, nor quadratic. If there exist a real number $w > 3$ and infinitely many quadratic numbers $\alpha$ such that*

$$|\xi - \alpha| < H(\alpha)^{-w},$$

*then $\xi$ is transcendental (as previously $H(\alpha)$ is the height of $\alpha$).*

### *9.1. Maillet–Baker's continued fractions*

Maillet proved in his already quoted book [58] that if $a = (a_n)_{n\geq 0}$ is a noneventually periodic sequence of positive integers, and if there are infinitely many positive integers $n$ such that

$$a_n = a_{n+1} = \cdots = a_{n+\lambda(n)},$$

then the real number $\xi = [a_0, a_1, a_2, \ldots]$ is transcendental provided that $\lambda(n)$ is larger than a certain function of $q_n$, the denominator of the $n$th convergent to $\xi$. Actually, the result of Maillet is more general and also includes the case of repetitions of a block of consecutive partial quotients. His proof is based on a general form of the Liouville inequality which limits the approximation of algebraic numbers by quadratic irrationals. Indeed, under the previous assumption, the quadratic irrational real numbers $\xi_n = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, a_n, \ldots, a_n, \ldots]$ provide approximations to $\xi$ that are "too good".

It is not very surprising that the breaktrough made by Roth [82] in 1955 led to an improvement of this result. Thus, Baker [12] used in 1962 the Roth theorem for number fields obtained by LeVeque [51] to strongly improve the results of Maillet and make them more explicit. His main idea was to see that if the quadratic approximations found by Maillet lie in a same quadratic number field, then one can favourably replace the use of the Liouville inequality by LeVeque's result. Amongst the results in [12], Baker proved in particular the following theorem:

**Theorem 33** (*A. Baker*). *Let $A$ be a positive integer and $\xi := [a_0, a_1, a_2, \ldots]$ be a real number whose partial quotients are all bounded by $A$. Let us assume that there exist an increasing sequence of positive integers $(n_k)_{k\geq 1}$, and a sequence of positive integers $(\lambda_k)_{k\geq 1}$, such that*

$$a_{n_k} = a_{n_k+1} = \cdots = a_{n_k+\lambda_k}.$$

*If*

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} > 2\left(\frac{\log\left(\left(A+\sqrt{A^2+4}\right)/2\right)}{\log\left(\left(1+\sqrt{5}\right)/2\right)}\right) - 1,$$

*then the real number $\xi$ is transcendental.*

In order to improve this result, it is quite tempting to apply the Subspace Theorem instead of LeVeque's theorem. This general trend of ideas was introduced in 2002 by Corvaja and Zannier for other questions of Diophantine approximation, e.g., transcendence of values of lacunary series at algebraic points, ratio of recurrence sequences, integer points on an algebraic curve or surface... (see [30]). We first mention that a direct use of Theorem 32 in Baker's approach leads to a weaker form of Theorem 33, contrarily to what is claimed in [65]. It seems, however, possible to reach a smaller bound than the one obtained in Theorem 33 by using the quadratic approximations previously considered by Maillet or by Baker and the ideas of [2], see [40]. In [7], the authors show how the method introduced in Section 9.2 can also be used to improve Theorem 33 in some particular cases.

Quite surprisingly, a tricky use of the Subspace Theorem based on the mirror formula allows the authors of [5] to considerably relax the transcendence criteria obtained by Baker. In particular, they get rid of the bound $A$ of Theorem 33 by proving the following result:

**Theorem 34** (*Adamczewski and Bugeaud*). *In Theorem 33, the assumption*

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} > 2\left(\frac{\log\left(\left(A+\sqrt{A^2+4}\right)/2\right)}{\log\left(\left(1+\sqrt{5}\right)/2\right)}\right) - 1,$$

*can be replaced by the weaker condition*

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} > 0.$$

### *9.2. Palindromic continued fractions*

A common feature of the results mentioned at the beginning of this section is that they apply to real numbers whose continued fraction expansions are "quasi-periodic", in the sense that they contain arbitrarily long blocks of partial quotients which occur precociously at least twice. We now consider real numbers whose sequence of partial quotients enjoys another combinatorial property, namely is "symmetrical", in the sense that it begins in arbitrarily long palindromes or quasi-palindromes. The results stated below are proved in [7] (see also [4]) and rest on the Subspace Theorem.

We first mention the following simple transcendental criterion for palindromic continued fractions.

**Theorem 35** (*Adamczewski and Bugeaud*). *Let $a = (a_n)_{n\geq 0}$ be a sequence of positive integers. If the word a begins in arbitrarily long palindromes, then the real number $\xi := [a_0, a_1, \ldots, a_n, \ldots]$ is either quadratic or transcendental.*

As shown in [4], given two distinct positive integers $a$ and $b$, Theorem 35 easily implies the transcendence of the real number $[a_0, a_1, a_2, \ldots]$, whose sequence of partial quotients is the Thue–Morse sequence on the alphabet $\{a, b\}$, i.e., with $a_n := a$ (resp. $a_n := b$) if the sum of the binary digits of $n$ is odd (resp. even). This result is originally due to Queffélec [77] who used a different approach. We also point out that, quite surprisingly, there is no assumption on the growth of the sequence $(a_n)_{n\geq 0}$ in Theorem 35.

Let us introduce some more notation. In order to relax the "symmetry" property of palindromes, we now introduce the notion of *quasi-palindrome*. Let $Z$ be a finite word. We say that $Z$ is a *quasi-palindrome of finite order* if there exist two finite words $U$ and $V$ such that $Z = UV\overline{U}$. Following this definition, the larger the quotient $|V|/|U|$, the weaker the symmetry property. Note that any palindrome is a quasi-palindrome (where $V$ is either empty or reduced to a single letter). We now give a transcendence criterion in which occurrences of arbitrarily long palindromes are replaced by occurrences of arbitrarily long quasi-palindromes $Z = UV\overline{U}$ where the quotient $|V|/|U|$ is bounded.

An extra assumption on the growth of the partial quotients is then needed. This assumption is not very restrictive. In particular, it is always satisfied by real numbers with bounded partial quotients.

Let $a = (a_n)_{n\geq 0}$ be a sequence over $\mathcal{A}$. We say that $a$ begins in *arbitrarily long quasi-palindromes of bounded order* if there exist a nonnegative real number $w$, and two sequences of finite words $(U_k)_{k\geq 0}$ and $(V_k)_{k\geq 0}$ such that:

(i) For any $k \geq 0$, the word $U_k V_k \overline{U_k}$ is a prefix of the word $a$;
(ii) The sequence $(|V_k|/|U_k|)_{k\geq 0}$ is bounded from above by $w$;
(iii) The sequence $(|U_k|)_{k\geq 0}$ is increasing.

Then, Theorem 35 can be extended in the following way:

**Theorem 36** (*Adamczewski and Bugeaud*). *Let $a = (a_n)_{n\geq 0}$ be a sequence of positive integers. Let $(p_n/q_n)_{n\geq 0}$ denote the sequence of convergents to the real number*

$$\xi := [a_0, a_1, \ldots, a_n, \ldots].$$

*Assume that the sequence $(q_n^{1/n})_{n\geq 0}$ is bounded, which is in particular the case when the sequence $a$ is bounded. If $a$ begins in arbitrarily long quasi-palindromes of bounded order, then $\xi$ is either quadratic or transcendental.*

In the statements of Theorems 35 and 36 the palindromes or the quasi-palindromes must appear at the very beginning of the continued fraction under consideration. We mention that the ideas used in their proofs also allow to deal with the more general situation where arbitrarily long quasi-palindromes occur not too far from the beginning (see [7]).

## 10. The folding lemma. Formal Laurent series

We begin this section with a variation on the mirror formula called the *folding lemma* (see [69,73]), whose proof is an easy consequence of the matrix formalism for continued fractions and of the mirror formula.

**Lemma 37** (*Folding Lemma*). *Let $c, a_0, a_1, \ldots$ be positive integers. Let $\frac{p_n}{q_n} := [a_0, a_1, \ldots, a_n]$. Then*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{cq_n^2} = [a_0, a_1, a_2, \ldots, a_n, c, -a_n, -a_{n-1}, \ldots, -a_1]. \tag{14}$$

**Remark 38.** In equality (14) negative partial quotients occur. An easy transformation permits to get rid of these forbidden partial quotients (see, e.g., [73]). Note that the terminology "folding lemma" comes from the fact that, defining the word $W := a_1 a_2 \cdots a_n$ and noting $\overline{W} := a_n a_{n-1} \cdots a_1$, we go from $\frac{p_n}{q_n}$ to $\frac{p_n}{q_n} + \frac{(-1)^n}{cq_n^2}$ (up to the first partial quotient $a_0$) by means of the "perturbed symmetry" (see [62,20], see also [61, p. 209]) $W \longrightarrow W\ c\ \overline{(-W)}$: iterating this operation in the case $W := +1$ and $c := +1$ gives a sequence of $\pm 1$ symbols that is the sequence of creases in a strip of paper repeatedly folded in half: see for example [42]. Note that a systematic use of transforming forbidden (i.e., $\leq 0$) partial quotients into permitted (i.e., $> 0$) ones led van der Poorten to state his very useful *ripple lemma* [70, Proposition 3].

As previously mentioned formal Laurent series can be expanded into continued fractions whose partial quotients are polynomials. The mirror formula and the folding lemma still hold in this context. We only give here a theorem due to van der Poorten and Shallit (see [76], see also [52] from a remark of Shallit given in [73]).

**Theorem 39.** *Let $F$ be the formal Laurent series $F = F(X) := X \sum_{h\geq 0} X^{-2^h}$. Then its continued fraction expansion is equal to*

$$[1, X, -X, -X, -X, X, X, -X, \ldots]$$

*where the sequence of partial quotients starting from the first $X$ is obtained by repeatedly iterating the folding rule: $W_0 := X$, $W_{j+1} := W_j (-X) \overline{(-W_j)}$ for $j \geq 0$.*

**Remark 40.** As previously $\overline{W}$ stands for the reversal of $W$, so that $W_1 = X\ -X\ -X$, $W_2 = X\ -X\ -X\ -X\ X\ X\ -X$. Note that the same folding trick permitted to Shallit [89] and to Kmošek [50] independently to give the continued fraction expansion of real numbers with explicit $g$-adic expansion such as $\sum 2^{-2^n}$. Also note that van der Poorten studies precisely how and when continued fraction expansions of Laurent formal series can be "specialized" or "reduced" modulo a prime number (see [69,71,74] and note that the folding lemma is alluded to in [71]).

We end this section by citing [6] where the authors, using the mirror formula, prove a result about the Littlewood conjecture for Laurent power series which is the analogue of Theorem 29.

## 11. Conclusion

Several other beautiful results about symmetrical or palindromic patterns in continued fraction expansions can be found in the literature: we refer the reader in particular to papers of van der Poorten, Tamura, Liardet–Stambul, Berstel–de Luca... ([72,73,75,92–94,53,17]...).

We do not resist ending this survey by citing two very nice papers on palindromes and continued fractions: one by Burger [22], who studies when a real quadratic irrational is a linear fractional transformation of its conjugate, the other by Benjamin and Zeilberger [15] who prove, revisiting a paper of Smith [91], that any prime congruent to 1 modulo 4 is the sum of two squares: the mirror formula and palindromic continued fraction expansions already present in Smith's paper are quite unexpected in this context.


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