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On powers of words occurring in binary codings of rotations

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Abstract

We discuss combinatorial properties of a class of binary sequences generalizing Sturmian sequences and obtained as a coding of an irrational rotation on the circle with respect to a partition in two intervals. We give a characterization of those having a finite index in terms of a two-dimensional continued fraction like algorithm, the so-called $\mathcal{D}$-expansion. Then, we discuss powers occurring at the beginning of these words and we prove, contrary to the Sturmian case, the existence of such sequences without any non-trivial asymptotic initial repetition. We also show that any characteristic sequence (that is, obtained as the coding of the orbit of the origin) has non-trivial repetitions not too far from the beginning and we apply this property to obtain the transcendence of the continued fractions whose partial quotients arises from such sequences when the two letters are replaced by distinct positive integers.
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1. Introduction

It seems that the question of the repetition of finite words occurring in an infinite sequence was first looked by A. Thue [33] in 1906. It was then presented as a purely combinatorial exercise without any particular application. Today, the situation is quite different since this problematic has been related to various fields, including the transcendence of

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real numbers, Diophantine approximation and quasicrystals *via* the study of the spectrum of discrete Schrödinger operators. Numerous recent papers deal with these interactions, as for instance [5,8,11,16,18,21]. It thus is not surprising in view of the fascination exercised by Sturmian sequences since the seminal papers of M. Morse and G.A. Hedlund [24,25] that the question of repetition of words occurring in Sturmian words has been considered by many authors [5,9,10,13,14,18,19,22,23,34]. In order to study this problem, an outstanding result gives an interpretation of Sturmian sequences in terms of substitutions (also called S-adic expansion) and of the continued fraction expansion of the associated slope (see for instance [7] and also a more precise result in [6]).

Theorem 1. *Let* $\mathbf{u}$ *be a characteristic Sturmian sequence with slope* α. *If the continued fraction expansion of* α *is* $[0; a_1+1, a_2, \dots a_n, \dots]$, *then*

$$\mathbf{u} = \lim_{n\to\infty} \tau_1^{a_1} \circ \tau_2^{a_2} \dots \tau_1^{a_{2n+1}}(1),$$

where substitutions τ_1 *and* τ_2 *are defined by*

$$\begin{array}{ccc} & \tau_1 & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 21 \end{array} \quad \textit{and} \quad \begin{array}{ccc} & \tau_2 & \\ 1 & \longmapsto & 12\,. \\ 2 & \longmapsto & 2 \end{array}$$

More generally, most of the results concerning Sturmian words can be proved by using such an expression.

In a previous paper [2], we give a similar expression for binary codings of rotations in which the role played by the continued fraction expansion is replaced in a natural way by the so-called $\mathcal{D}$-expansion. Then, we use this representation in [1–4] to deduce different dynamical and combinatorial properties for these sequences and to deal with related questions arising at once from number theory, theoretical computer science and theoretical physic. In the present paper, we discuss the problem of powers of words occurring in this class of binary sequences as well as a related question concerning the transcendence of real numbers defined by their continued fraction expansions. In this direction, it has been for instance proved in [5] that the real number $\alpha := [0, 1, 2, 1, 1, 2, 1, 2, \dots]$ having the Fibonacci sequence as continued fraction expansion is transcendental. More recently, D. Roy [29–31] (see also [12]) exhibited surprising properties for similar numbers related to an old conjecture in the study of the approximation of real numbers by algebraic integers. The attractive work of D. Roy should motive further development in this area of mathematics.

2. Definitions

2.1. Binary codings of rotations

Let $(\alpha, \beta) \in (0,1)^2$. In the following $\{x\}$ will mean the fractional part of the real x. The coding of rotation corresponding to the parameters (α, β, x) is defined as the symbolic sequence $\mathbf{u} = (u_n)_{n\geqslant 0}$ with value in the binary alphabet $\{0, 1\}$ by:

$$u_n = \begin{cases} 1 & \text{if } \{x + n\alpha\} \in [0, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

When α is rational, the sequences obtained are clearly periodic, otherwise the coding of rotation is said irrational. The cases $\beta = \alpha$ or $\beta = 1 - \alpha$ give Sturmian sequences and, more generally, the case $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$ gives quasi-Sturmian sequences (see [28] and see Section 4 for a definition of Sturmian and quasi-Sturmian sequences). In all that follows, we only consider irrational binary codings of rotations. A binary coding of rotation is said to be non-degenerate if $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$. A coding of rotation is called characteristic if $x = 0$.

2.2. *Interval exchange transformations*

Let $s \in \mathbb{N}$, $s \geqslant 2$. Let σ be a permutation of the set $\{1, 2, \dots, s\}$ and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ be a vector in $\mathbb{R}^s$ with positive entries. Let

$$I = \big[0, |\lambda|\big[, \quad \text{where } |\lambda| = \sum_{i=1}^{s} \lambda_i, \quad \text{and} \quad \text{for } 1 \leqslant i \leqslant s, \quad I_i = \bigg[\sum_{j<i} \lambda_j, \sum_{j \leqslant i} \lambda_j\bigg).$$

The interval exchange transformation associated with (λ, σ) is the map E from I into itself, defined as the piecewise isometry which arises from ordering the intervals I_i with respect to σ. More precisely, if $x \in I_i$,

$$E(x) = x + a_i, \quad \text{where } a_i = \sum_{k<\sigma^{-1}(i)} \lambda_{\sigma(k)} - \sum_{k<i} \lambda_k.$$

We can introduce a natural coding of the orbit of a point under the action of an interval exchange transformation by assigning to each element of this orbit the index of the interval which contains it. Such a coding is said characteristic if the starting point is the origin. We say that an interval exchange transformation satisfies the i.d.o.c. (infinite distinct orbit condition) introduced in [20] if the orbits of its discontinuities are infinite and distinct.

2.3. *The $\mathcal{D}$-expansion*

In a previous paper [2], we have investigated the links between non-degenerate binary codings of rotations and i.d.o.c. three-interval exchange transformations. An algorithm was introduced which can be regarded as a speed-up of the Rauzy induction for three-interval exchange transformations. This algorithm is also a generalization of the classical continued fraction algorithm. Let us introduce a map

$$\mathcal{D} : [0, 1) \times \mathbb{R}_+^* \longrightarrow [0, 1) \times \mathbb{R}_+^*$$

given by

$$(x, y) \longmapsto \begin{cases} \left(\dfrac{\left\{\frac{x}{y-1}\right\}}{\frac{1}{y-1} - \left\lfloor \frac{x}{y-1} \right\rfloor}, \dfrac{1}{\frac{1}{y-1} - \left\lfloor \frac{x}{y-1} \right\rfloor} \right) & \text{if } y > 1, \\ \left(\left\{ \dfrac{x}{1-y} \right\}, \dfrac{y}{1-y} - \left\lfloor \dfrac{x}{1-y} \right\rfloor \right) & \text{if } y < 1, \\ (0, 1) & \text{if } y = 1. \end{cases}$$

Given a non-degenerate coding of rotation with parameters (α, β), $\beta > \alpha$, the associated $\mathcal{D}$-expansion is given by the sequence $(a_n, i_n)_{n\in\mathbb{N}}$ which is defined as follows:

$$a_n = \left\lfloor \left| \frac{x_n}{y_n - 1} \right| \right\rfloor, \qquad i_n = \begin{cases} 1 & \text{if } y_n < 1, \\ 0 & \text{if } y_n > 1, \end{cases}$$

where $(x_n, y_n) = \mathcal{D}^n(x_0, y_0)$ and

$$(x_0, y_0) = \left(\frac{1 - \lfloor \frac{1-\beta}{\alpha} \rfloor \alpha - \beta}{1 - (\lfloor \frac{1-\beta}{\alpha} \rfloor + 1)\alpha}, \frac{\alpha}{1 - (\lfloor \frac{1-\beta}{\alpha} \rfloor + 1)\alpha} \right).$$

For a non-degenerate coding of rotation corresponding to $(\alpha, \beta) \in [0,1)^2$, $\alpha > \beta$, its $\mathcal{D}$-expansion is given by the $\mathcal{D}$-expansion associated with the coding corresponding to $(1-\alpha, 1-\beta)$. The $\mathcal{D}$-expansion of a non-degenerate coding of rotation always satisfies that $(a_n)_{n\in\mathbb{N}}$ is not eventually vanishing and that $(i_n)_{n\in\mathbb{N}}$ is not eventually constant.

2.4. Index of an infinite word and initial critical exponent

Let w be a finite word and p a positive integer. We will denote by w^p the word

$$w^p = \underbrace{ww\ldots w}_{p \text{ times}}.$$

More generally, when x is a positive real number w^x is the word $w^{\lfloor x \rfloor}u$, where u is the prefix of w with length $\lceil (x - \lfloor x \rfloor)|\omega| \rceil$. Let $\mathbf{u}$ be an infinite word. The index of $\mathbf{u}$, denoted by $\operatorname{ind}(\mathbf{u})$, is defined by:

$$\operatorname{ind}(\mathbf{u}) = \sup\{x \in \mathbb{R}\colon \exists w,\ w^x \in \mathcal{L}(\mathbf{u})\}.$$

We can also define the asymptotic index of $\mathbf{u}$, denoted by $\operatorname{ind}^*(\mathbf{u})$, by

$$\operatorname{ind}^*(\mathbf{u}) = \sup\{x \in \mathbb{R}\colon \forall n > 0, \exists w,\ |w| > n,\ w^x \in \mathcal{L}(\mathbf{u})\}.$$

These definitions easily imply that $\operatorname{ind}(\mathbf{u})$ and $\operatorname{ind}^*(\mathbf{u})$ are together finite or infinite. In the case where these quantities are both finite, we will say that $\mathbf{u}$ has a finite index.

Let p be a positive integer. A word $\mathbf{u}$ is said p-power free if $w^p \in \mathcal{L}(\mathbf{u}) \Rightarrow w = \varepsilon$ and asymptotically p-power free if there exists an integer N such that $(w^p \in \mathcal{L}(\mathbf{u})$ and $|w| > N) \Rightarrow w = \varepsilon$.

In the remainder of the paper, we will write $u \prec w$ when the word u is a prefix of the word w. We define the initial critical exponent of an infinite word $\mathbf{u}$ by

$$\operatorname{ice}(\mathbf{u}) = \sup\{x \in \mathbb{R}\colon \forall n > 0, \exists w,\ |w| > n,\ w^x \prec \mathbf{u}\}.$$

This definition obviously implies that $\operatorname{ice}(\mathbf{u}) \geqslant 1$ for any sequence $\mathbf{u}$ and we will exhibit in the following examples of sequences without any asymptotic initial power, that is, for which ice is equal to 1.

3. Main results

We present here our three main theorems (that is, Theorems 3, 8, and 10) together with related results already obtained in the case of Sturmian sequences.

3.1. Words with a finite index

The following characterization of Sturmian sequences with a finite index was first proved by P. Mignosi in [22] and the quantitative part of this result is due more recently to D. Vandeth [34].

Theorem 2 (Mignosi, Vandeth). *Let* $\mathbf{u}$ *be a Sturmian sequence of slope* α *and let* $(a_n)_{n\geqslant 0}$ *be its continued fraction expansion. Then* $\mathbf{u}$ *has a finite index if and only if* α *has bounded partial quotients.*

Moreover if α *has bounded partial quotients, then* $\mathbf{u}$ *is asymptotically* k*th power-free but not asymptotically* $(k-1)$*th power-free with* $k = 3 + \limsup\{a_n\colon\ n \geqslant 0\}$.

The following result proved in Section 5 is an analogous of Theorem 2 in the case of non-degenerate binary codings of rotations, the $\mathcal{D}$-expansion taking the place of the classical continued fraction expansion.

Theorem 3. *Let* $\mathbf{u}$ *be a non-degenerate binary coding of rotation and let* $(a_n, i_n)_{n\geqslant 0}$ *be its* $\mathcal{D}$*-expansion. Then,* $\mathbf{u}$ *has a finite index if and only if there exists a non-negative integer* M *satisfying*:

(i) $a_n \leqslant M$,
(ii) $i_n = i_{n+1} = \cdots = i_{n+M-1} \Rightarrow \exists k,\ n \leqslant k \leqslant n+M-1$ *such that* $a_k \neq 0$.

Moreover, if there exist infinitely many n *such that either* $a_n = M$ *or such that* $i_n = i_{n+1} = \cdots = i_{n+M-2}$ *and* $a_k = 0$, $n \leqslant k \leqslant n+M-2$, *then* $\mathbf{u}$ *is asymptotically* k*th power-free but not asymptotically* $(k-3)$*th power-free for* $k = M+4$.

A characterization of the non-degenerate binary codings of rotations which are linearly recurrent is given again in terms of their $\mathcal{D}$-expansion in [4]. This result together with Theorem 3 allows us to characterize the codings of rotations with a finite index and which are not linearly recurrent. Let us note that a Sturmian sequence has a finite index if and only if it is linearly recurrent.

Corollary 4. *Let* $\mathbf{u}$ *be a binary coding of rotation and let* $(a_n, i_n)_{n\geqslant 0}$ *be its* $\mathcal{D}$*-expansion. Then,* $\mathbf{u}$ *has a finite index but is not linearly recurrent if and only if there exists a non-negative integer* M *satisfying*:

(i) $a_n \leqslant M$,
(ii) $i_n = i_{n+1} = \cdots = i_{n+M-1} \Rightarrow \exists k,\ n \leqslant k \leqslant n+M-1$ *such that* $a_k \neq 0$,
(iii) $\forall n > 0,\ \exists k$ *such that* $a_{k+j} = 0$, $0 \leqslant j \leqslant n$.

3.2. Words without any non-trivial asymptotic initial repetition

The study of arbitrarily long initial powers in infinite words has recently been related to important questions in different areas of mathematics and theoretical physic. The following combinatorial condition of transcendence is proved under the lines in [5]. Their proof is based on a famous result of W.M. Schmidt [32] on simultaneous approximation by algebraic numbers. The method they used was first initiated in [15] and in [26,27].

Theorem 5 (Allouche et al.). *Let $\mathbf{u} = (u_n)_{n\geqslant 0}$ be a sequence of positive integers taking two different values. If the subshift generated by $\mathbf{u}$ is uniquely ergodic and if* $\mathrm{ice}(\mathbf{u}) > 3/2$, *then the real number $\alpha := [0, u_0, u_1, \ldots, u_n, \ldots]$ is transcendental.*

In particular, this condition is used again in [5] to prove the transcendence of Sturmian continued fractions.

Theorem 6 (Allouche et al.). *Let $\mathbf{u} = (u_n)_{n\geqslant 0}$ be a sequence of positive integers. Then the real number $\alpha := [0, u_0, u_1, \ldots, u_n, \ldots]$ is transcendental if the sequence $\mathbf{u}$ is quasi-Sturmian* (*the definition of quasi-Sturmian sequences is given in Section* 4). *In particular, the same conclusion holds when $\mathbf{u}$ is a Sturmian sequence.*

The key point to prove the transcendence of Sturmian continued fractions and also to study the spectrum of one-dimensional discrete Schrödinger operators with Sturmian sequences as potentials in [13] is given by the following property of Sturmian sequences obtained independently in [5,13].

Theorem 7. *Any Sturmian sequence begins in arbitrarily long squares.*

We will show in Section 6 as a consequence of the following result that such a nice property is generally far to be true for binary codings of rotations.

Theorem 8. *There exist a non-countable number of characteristic binary codings of rotations with initial critical exponent equal to* 1.

However, we can easily prove by using Theorem 11 and a result of W. Veech [35, p. 225] that the presence of arbitrarily long squares at the beginning of characteristic codings of rotations is generic.

Proposition 9. *For almost all parameters (α, β)* (*in the sense of the Lebesgue measure on the square*), *the corresponding characteristic coding of rotation begins in arbitrarily long squares.*

More precisely, such a conclusion holds if there are infinitely many positive integers n such that one of the following holds:

$$\text{(i)}\quad a_n = i_n = 0, \qquad \text{(ii)}\quad i_n = i_{n+1} = 0,$$

where $(a_n, i_n)_{n\geqslant 0}$ denotes the $\mathcal{D}$-expansion associated with (α, β).

3.3. Almost initial powers and an application to transcendence

Though Theorem 8 shows the impossibility of applying Theorem 5, we will prove that characteristic codings of rotations always have non-trivial asymptotic repetitions not too far from the beginning (see Section 7 for a definition) and we will use this property together with a recent result of J.L. Davison [16] to obtain the following extension of Theorem 6.

Theorem 10. *Let* $\mathbf{u} = (u_n)_{n \geqslant 0}$ *be a sequence of positive integers. Then the number* $\alpha := [0, u_0, u_1, \ldots, u_n, \ldots]$ *is transcendental if one of the following holds*:

(i) *the sequence* $\mathbf{u}$ *is an irrational characteristic coding of rotation*;
(ii) *the sequence* $\mathbf{u}$ *is a characteristic coding of a non-periodic three-interval exchange transformation.*

The combinatorial condition of transcendence given in [16] generalizes Theorem 5 and is recalled in Section 7 (Theorem 30). Unfortunately, our method does not allow to prove the transcendence of the continued fractions associated to non-characteristic codings of rotations. This is mainly due to the fact that an explicit S-adic expression is just known for the characteristic sequences.

The organization of the article is as follows. In Section 4 we recall some classical definitions on infinite words and morphisms together with the S-adic expression obtained in [2] for non-degenerated binary codings of rotations. This representation in terms of substitutions will be the key tool in the remainder of the paper for proving our results. Sections 5–7 are devoted respectively to the proofs of Theorems 3, 8, and 10.

4. Background on infinite words

4.1. Sequences and morphisms

A finite word on $\mathcal{A}$ is a finite sequence of letters and an infinite word on $\mathcal{A}$ is a sequence of letters indexed by $\mathbb{N}$. The length of a finite word ω, denoted by $|\omega|$, is the number of letters it is built from. The empty word, ε, is the unique word of length 0. We denote by $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$ and by $\mathcal{A}^{\mathbb{N}}$ the set of sequences on $\mathcal{A}$.

Let $\mathbf{u} = (u_k)_{k \in \mathbb{N}}$ be a symbolic sequence defined over the alphabet $\mathcal{A}$. A factor of $\mathbf{u}$ is a finite word of the form $u_i u_{i+1} \ldots u_j$, $0 \leqslant i \leqslant j$. If ω is a factor of $\mathbf{u}$ and a a letter, then $|w|_a$ is the number of occurrences of the letter a in the word ω. We denote by $\mathcal{L}(\mathbf{u})$ the set of all the factors of the sequence $\mathbf{u}$. The set $\mathcal{L}(\mathbf{u})$ is called the language of $\mathbf{u}$. A sequence in which all the factors have an infinite number of occurrences is called recurrent. When these occurrences have bounded gaps, the sequence is called uniformly recurrent. A sequence is said linearly recurrent (with constant K) if there exists an integer K such that for any of its factors ω the difference between two successive occurrences is bounded by $K|\omega|$.

We call complexity function of a finitely-valued sequence $\mathbf{u}$ the function which associates with each integer n the number $p(n)$ of different words of length n occurring in $\mathbf{u}$.

A sequence is called Sturmian if $p(n) = n + 1$ for every integer n. More generally, a non-periodic sequence is called quasi-Sturmian if there exists a positive integer c such that $p(n) \leqslant n + c$ for every integer n.

Let $\mathbf{u}$ be a uniformly recurrent sequence over the alphabet $\mathcal{A}$ and let w be a nonempty prefix of $\mathbf{u}$. A return word to w of $\mathbf{u}$ is a factor $u_{[i,k-1]}$ $(= u_i \ldots u_{k-1})$ of $\mathbf{u}$ such that i and k are two consecutive occurrences of w. The sequence $\mathbf{u}$ can be written in a unique way as a concatenation of return words to u. Let $\mathcal{R}_{\mathbf{u},w}$ be the set of return words to w in $\mathbf{u}$. Then $\mathbf{u} = v_0 v_1 \ldots v_i \ldots$, where $v_i \in \mathcal{R}_{\mathbf{u},w}$. The fact that $\mathbf{u}$ is uniformly recurrent implies that $\mathcal{R}_{\mathbf{u},w}$ is a finite set. We can therefore consider a bijective map $\Lambda_{\mathbf{u},w}$ from $\mathcal{R}_{\mathbf{u},w}$ to the finite set $\{1, 2, \ldots, \mathrm{Card}(\mathcal{R}_{\mathbf{u},w})\}$, where, for definiteness, the return words are ordered according to their first occurrence (i.e., $\Lambda^{-1}_{\mathbf{u},w}(1)$ is the first return word v_0, $\Lambda^{-1}_{\mathbf{u},w}(2)$ is the first v_i which is different from v_0, and so on). The derived sequence of $\mathbf{u}$ on w is the sequence with values in the alphabet $\{1, 2, \ldots, \mathrm{Card}(\mathcal{R}_{\mathbf{u},w})\}$ given by

$$\mathcal{D}_w(\mathbf{u}) = \Lambda_{\mathbf{u},w}(v_0)\Lambda_{\mathbf{u},w}(v_1)\ldots\Lambda_{\mathbf{u},w}(v_i)\ldots.$$

To such a sequence we can associate a morphism ϕ_w by

$$\phi_w(i) = v_i.$$

We obtain $\phi_w(\mathcal{D}_w(\mathbf{u})) = \mathbf{u}$. The morphism ϕ_w is called the return morphism to w of $\mathbf{u}$.

In the following, morphism will mean homomorphism of (free) monoid. A morphism ϕ such that no letter is mapped to the empty word is said to be non-erasing. A non-erasing endomorphism of free monoid is called substitution. All the morphisms considered in the remainder of the paper will be non-erasing.

4.2. *An S-adic representation of binary codings of rotations*

Let k be a non-negative integer. Let us consider the two following substitutions defined by

$$\begin{array}{rcl} & \mathcal{F}_k & \\ 1 & \longmapsto & 13 \\ 2 & \longmapsto & 2^{k+1}3 \\ 3 & \longmapsto & 2^k 3 \end{array} \quad \text{and} \quad \begin{array}{rcl} & \mathcal{G}_k & \\ 1 & \longmapsto & 12^k \\ 2 & \longmapsto & 12^{k+1} \\ 3 & \longmapsto & 13 \end{array} \tag{1}$$

and let ϕ_k be the morphism defined from $\{1, 2, 3\}$ to $\{1, 2\}$ by

$$\begin{array}{rcl} & \phi_k & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 12^{k+1} \\ 3 & \longmapsto & 12^k. \end{array}$$

With such a notation, we have given in [2] the following S-adic representation of characteristic non-degenerate codings of rotations.

Theorem 11. *Let* $\mathbf{u}$ *be a characteristic non-degenerate coding of rotation with* $\mathcal{D}$*-expansion equal to* $(a_n, i_n)_{n\geqslant 0} \in (\mathbb{N} \times \{0,1\})^{\mathbb{N}}$ *and let* $\mathbf{v}$ *be defined by*

$$\mathbf{v} = \lim_{n\to\infty} \left(\prod_{j=0}^{n} \left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)(1) \right),$$

where $\prod$ *means the composition of morphisms from left to right. Then,* $\mathbf{v}$ *is the natural coding of an i.d.o.c. three-interval exchange and there exists an explicit non-erasing morphism* ϕ *from* $\{1,2,3\}$ *to* $\{1,2\}$ *such that either* $\mathbf{u} = \phi(\mathbf{v})$ *or* $\mathbf{u} = 1S(\phi(\mathbf{v}))$*, where* S *denotes the classical shift transformation.*

Conversely, the following result is particularly useful to exhibit codings of rotations or of three-interval exchanges with a prescribed property. We will use it for instance in Section 6 for proving Theorem 8.

Theorem 12. *Let* k *be a non-negative integer and let* $(a_n, i_n)_{n\geqslant 0} \in (\mathbb{N} \times \{0,1\})^{\mathbb{N}}$*, with* $(a_n)_{n\geqslant 0}$ *not eventually vanishing and with* $(i_n)_{n\geqslant 0}$ *not eventually constant. Then, the sequence*

$$\mathbf{v} = \lim_{n\to\infty} \left(\prod_{j=0}^{n} \left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)(1) \right),$$

is a natural coding of a three-interval exchange transformation and the sequence $\mathbf{u} = \phi_k(\mathbf{v})$ *is a characteristic coding of rotation having the sequence* $(a_n, i_n)_{n\geqslant 0}$ *as* $\mathcal{D}$*-expansion.*

5. Words with a finite index: proof of Theorem 3

Our proof of Theorem 3 follows an idea introduced in [4]. In the remainder of this section, M denotes a fixed non-negative integer. We first have to prove that any infinite sequence $\mathbf{v}$ defined as in Theorem 11 is $(M+3)$th power-free when the sequence $(a_n, i_n)_{n\geqslant 0}$ satisfies the conditions (i) and (ii) of Theorem 3. In order to prove this point, we define three language over $\{1,2,3\}$ which does not contain any $(M+3)$th power and with extra conditions. Then, we obtain stability properties (see Lemmas 14, 16–18) for the image of our languages by the different substitutions ($\mathcal{F}_k$, $\mathcal{G}_k$, $\mathcal{F}_0^k$, and $\mathcal{G}_0^k$) used to built the sequence $\mathbf{v}$. These properties allow us to show that at each step of the construction of the sequence $\mathbf{v}$, the finite prefix that we obtain does not have a $(M+3)$th power as a factor, proving that $\mathbf{v}$ is $(M+3)$th power-free. The fact that $\mathbf{u}$ is asymptotically $(M+3)$th-power free finally derives from a classical trick on derived sequences.

Let $\mathcal{L}_1$ be the largest language (with respect to the inclusion) defined over the alphabet $\{1,2,3\}$ and satisfying the following conditions:

(i) $\forall \omega \in \{1,2,3\}^*$, $\omega^{M+3} \in \mathcal{L}_1 \Rightarrow \omega = \varepsilon$,
(ii) $\forall \omega \in \{1,2,3\}^*$ and $\forall z \in \{1,2,3\}$, $(\omega z)^{M+2}\omega \notin \mathcal{L}_1$,

(iii) $33 \in \mathcal{L}_1 \Rightarrow (21 \notin \mathcal{L}_1$ and $22 \notin \mathcal{L}_1)$,
(iv) $11 \in \mathcal{L}_1 \Rightarrow (22 \notin \mathcal{L}_1$ and $23 \notin \mathcal{L}_1)$.

Let $\mathcal{L}_2$ be the largest language defined over the alphabet $\{1,2,3\}$ and satisfying the following conditions:

(i) $\forall \omega \in \{1,2,3\}^*$, $\omega^{M+3} \in \mathcal{L}_2 \Rightarrow \omega = \varepsilon$,
(ii) $\forall \omega \in \{1,2,3\}^*$ and $\forall z \in \{1,2,3\}$, $(\omega z)^{M+2}\omega \notin \mathcal{L}_2$,
(iii) $33 \notin \mathcal{L}_2$.

Let $\mathcal{L}_3$ be the largest language defined over the alphabet $\{1,2,3\}$ and satisfying the following conditions:

(i) $\forall \omega \in \{1,2,3\}^*$, $\omega^{M+3} \in \mathcal{L}_3 \Rightarrow \omega = \varepsilon$,
(ii) $\forall \omega \in \{1,2,3\}^*$ and $\forall z \in \{1,2,3\}$, $(\omega z)^{M+2}\omega \notin \mathcal{L}_3$,
(iii) $11 \notin \mathcal{L}_3$.

These three languages are clearly all included in the language $\mathcal{L}$ defined over the alphabet $\{1,2,3\}$ as the largest one satisfying the condition:

$$\forall \omega \in \{1,2,3\}^*, \quad \omega^{M+3} \in \mathcal{L} \quad \Rightarrow \quad \omega = \varepsilon.$$

In the remainder of this section, $\mathcal{F}_k$ and $\mathcal{G}_k$ denote the substitutions defined in (1).

Lemma 13. *Let k be a non-negative integer, $w \in \mathcal{F}_k(\mathcal{L}_1)$ and $z \in \{1,2\}$. Then, if $3w\mathcal{F}_k(z) \in \mathcal{F}_k(\mathcal{L}_1)$ there exists a word $w' \in \{1,2,3\}^*$ such that $w = \mathcal{F}_k(w')$.*

Proof. The proof directly comes from the definition of $\mathcal{F}_k$. □

Lemma 14. *Let k be a positive integer, $k \leqslant M$. Then*

$$\mathcal{F}_k(\mathcal{L}_1) \subset \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3.$$

Proof. Let k be a positive integer, $k \leqslant M$. The fact that $k \geqslant 1$ implies that the letters 1 and 3 are isolated in $\mathcal{F}_k(\mathcal{L}_1)$. Thus, $\mathcal{F}_k(\mathcal{L}_1)$ satisfies the conditions (iii) and (iv) in the definition of $\mathcal{L}_1$ and the condition (iii) in the definitions of $\mathcal{L}_2$ and $\mathcal{L}_3$.

It remains to prove that the conditions (i) and (ii) in the definition of $\mathcal{L}_1$ are satisfied by $\mathcal{F}_k(\mathcal{L}_1)$.

(i) Let $\omega \in \{1,2,3\}^*$ such that $\omega^{M+3} \in \mathcal{F}_k(\mathcal{L}_1)$. Then there exists a 3-tuple of finite words (x, v, y), $x \in \{\varepsilon, 2^l3\colon 1 \leqslant l \leqslant k\}$, $v \in \{1,2,3\}^*$, $y \in \{\varepsilon, 1, 2^l\colon 1 \leqslant l \leqslant k+1\}$, such that

$$\omega = x\mathcal{F}_k(v)y. \tag{2}$$

Moreover, the unicity of the decomposition is ensured if v is chosen with a maximal length and if 3 is not a prefix of v. Let us assume that ω is a non-empty word. Two cases have to be considered.

(a) If $v = \varepsilon$ then $(xy)^{M+3} \in \mathcal{F}_k(\mathcal{L}_1)$. Since $|yx|_3 \leqslant 1$, this should imply the existence of a letter z such that $z^{M+2} \in \mathcal{L}_1$. A contradiction appears since z^{M+2} cannot lie in $\mathcal{L}_1$ in view of (ii).

(b) Let us assume that $v \neq \varepsilon$. Following the decomposition given in (2), we obtain that $\omega^{M+3} = x(\mathcal{F}_k(v)yx)^{M+2}\mathcal{F}_k(v)y$. Since v does not begin with a 3 and that $|yx|_3 \leqslant 1$, Lemma 13 implies the existence of a letter $a \in \{1, 2, 3\}$ such that $yx = \mathcal{F}_k(a)$. We thus have

$$\omega^{M+3} = x\bigl(\mathcal{F}_k(va)\bigr)^{M+2}\mathcal{F}_k(v)y = x\mathcal{F}_k\bigl((va)^{M+2}v\bigr)y.$$

The fact that v does not begin with 3 implies that $(va)^{M+2}v \in \mathcal{L}_1$, which gives a contradiction with (ii). In fact, if we consider a word $u \in \{1, 2, 3\}^*$ such that $\mathcal{F}_k(u) \in \mathcal{F}_k(\mathcal{L}_1)$ and u does not begin with 3, then $u \in \mathcal{L}_1$.

(ii) Let $\omega \in \{1, 2, 3\}^*$ and $z \in \{1, 2, 3\}$ such that $(\omega z)^{M+2}\omega \in \mathcal{F}_k(\mathcal{L}_1)$. Let us consider the decomposition of ω in $\omega = x\mathcal{F}_k(v)y$ (as in (2)). Thus $(\omega z)^{M+2}\omega = x(\mathcal{F}_k(v)yzx)^{M+2}\mathcal{F}_k(v)y$. Since v does not begin with 3 and $|yzx|_3 \leqslant 2$, Lemma 13 implies the existence of a letter a such that $yzx = \mathcal{F}_k(a)$ or the existence of two letters a and b such that $yzx = \mathcal{F}_k(ab)$.

(a) If $yzx = \mathcal{F}_k(a)$, then $(\omega z)^{M+2}\omega = x\mathcal{F}_k((va)^{M+2}v)y$. This would imply that $(va)^{M+2}v \in \mathcal{L}_1$ because v does not begin with 3, which will give a contradiction with (ii).

(b) If $yzx = \mathcal{F}_k(ab)$, then $z = 3$, $b = 3$ and $x = 2^k3$ because $x \in \{\varepsilon, 2^l3\colon 1 \leqslant l \leqslant k\}$ and $y \in \{\varepsilon, 1, 2^l\colon 1 \leqslant l \leqslant k+1\}$. It thus follows $(\omega z)^{M+2}\omega = 2^k3\mathcal{F}_k((va3)^{M+2}v)y$.
   - (b_1) If $a = 1$, then $y = 1$ and $(\omega z)^{M+2}\omega = 2^k3\mathcal{F}_k((v13)^{M+2}v)13 \in \mathcal{F}_k(\mathcal{L}_1)$ since the letter 1 is always followed with 3; so $(\omega z)^{M+2}\omega = 2^k3\mathcal{F}_k((v13)^{M+2}v1) \in \mathcal{F}_k(\mathcal{L}_1)$. Since v does not begin with 3, $(v13)^{M+2}v1 \in \mathcal{F}_k(\mathcal{L}_1)$, which gives a contradiction with (ii).
   - (b_2) If $a = 2$, then $y = 2^{k+1}$ and we obtain that $(\omega z)^{M+2}\omega = 2^k3\mathcal{F}_k((v23)^{M+2}v2) \in \mathcal{F}_k(\mathcal{L}_1)$ because 2^{k+1} is always followed with 3. Since v does not begin with 3, it follows $(v23)^{M+2}v2 \in \mathcal{F}_k(\mathcal{L}_1)$, which gives a contradiction with (ii).
   - (b_3) If $a = 3$ and since $x = 2^k3$ then either $3(v33)^{M+2}v \in \mathcal{L}_1$ or $2(v33)^{M+2}v \in \mathcal{L}_1$. The first case gives a contradiction with (ii) because $3(v33)^{M+2}v = (3v3)^{M+2}3v$. In the last case and since v does not begin with 3, $2v$ should have 21 or 22 as a factor. But since 33 should also be a factor of $\mathcal{L}_1$, this gives a contradiction with (iii). □

Lemma 15. *Let k be a non-negative integer, $w \in \mathcal{G}_k(\mathcal{L}_1)$ and $z \in \{2, 3\}$. Then, if $\mathcal{F}_k(z)w1 \in \mathcal{F}_k(\mathcal{L}_1)$ there exists a word $w' \in \{1, 2, 3\}^*$ such that $w = \mathcal{G}_k(w')$.*

Proof. It directly follows from the definition of the substitution $\mathcal{G}_k$. $\square$

Lemma 16. *Let k be a positive integer, $k \leqslant M$. Then*

$$\mathcal{G}_k(\mathcal{L}_1) \subset \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3.$$

Proof. This proof is similar to the one of Lemma 14. This comes from the symmetry between the substitutions $\mathcal{F}_k$ and $\mathcal{G}_k$, for which the role of the letters 1 and 3 are just exchanged. The central place of the letter 3 in the proof of Lemma 14 is now given to the letter 1. In order to help the reader, we recall for words in $\mathcal{G}_k(\mathcal{L}_1)$ an analogous decomposition of the one given in (2). Let $\omega \in \{1,2,3\}^*$ such that $\omega^{M+3} \in \mathcal{G}_k(\mathcal{L}_1)$. Then, there exists a 3-tuple of words (x, v, y), $x \in \{\varepsilon, 3, 2^l\colon\ 1 \leqslant l \leqslant k+1\}$, $v \in \{1,2,3\}^*$, $y \in \{\varepsilon, 12^l\colon\ 1 \leqslant l \leqslant k\}$, such that

$$\omega = x\mathcal{G}_k(v)y. \qquad \square$$

Lemma 17. *Let k be a positive integer, $k \leqslant M$. Then*

$$\mathcal{F}_0^k(\mathcal{L}_2) \subset \mathcal{L}_1 \cap \mathcal{L}_3.$$

Proof. We recall that the substitution $\mathcal{F}_0^k$ is defined by

$$\begin{array}{rcl} & \mathcal{F}_0^k & \\ 1 & \longmapsto & 13^k \\ 2 & \longmapsto & 23^k \\ 3 & \longmapsto & 3. \end{array}$$

This definition implies that none of the words 11, 21 or 22 lie in the language $\mathcal{F}_0^k(\mathcal{L}_2)$. In particular, $\mathcal{F}_0^k(\mathcal{L}_2)$ satisfies the assertions (iii) and (iv) in the definition of $\mathcal{L}_1$ and the assertion (iii) in the definition of $\mathcal{L}_3$.

Now, let us show that the assertions (i) and (ii) in the definitions of $\mathcal{L}_1$ and $\mathcal{L}_3$ are satisfied by $\mathcal{F}_0^k(\mathcal{L}_2)$.

(i) Let $\omega \in \{1,2,3\}^*$ such that $\omega^{M+3} \in \mathcal{F}_0^k(\mathcal{L}_2)$. Then, there exists a triple of words (x, v, y) with $x \in \{\varepsilon, 3^l\colon\ 1 \leqslant l \leqslant k+1\}$, $v \in \{1,2,3\}^*$ and $y \in \{\varepsilon, 1, 13^l, 23^l\colon\ 1 \leqslant l \leqslant k+1\}$, such that

$$\omega = x\mathcal{F}_k(v)y. \tag{3}$$

Moreover, this decomposition is unique if v is chosen with a minimal length and if the letter 3 is not a prefix of v. Let us assume that ω is not reduced to the empty word. We thus have two cases to consider.

(a) If $v = \varepsilon$, then $(xy)^{M+3} \in \mathcal{F}_0^k(\mathcal{L}_2)$. If moreover $y = \varepsilon$, then we should have $3^{M+2} \in \mathcal{F}_0^k(\mathcal{L}_2)$, which is impossible because $33 \notin \mathcal{L}_2$. Since x cannot be the empty word, we obtain the existence of a letter $a \in \{1,2\}$ such that either $xy = \mathcal{F}_0^k(a)$ or $yx = \mathcal{F}_0^k(a3)$.

In the first case, we will obtain that $a^{M+2} \in \mathcal{L}_2$ since a is not equal to 3. This produces a contradiction with (ii). In the second case, the same reasoning gives $(a3)^{M+2}a \in \mathcal{L}_2$, hence again a contradiction with (ii).

(b) Now if $v \neq \varepsilon$, then ω^{M+3} can be decomposed as $\omega^{M+3} = x(\mathcal{F}_0^k(v)yx)^{M+2}\mathcal{F}_0^k(v)y$. Since v does not begin with 3 and since $|yx|_1 + |yx|_2 \leqslant 1$, Lemma 13 implies the existence of a letter $a \in \{1,2,3\}$ such that either $yx = \mathcal{F}_0^k(a) \in \mathcal{F}_0^k(\mathcal{L}_2)$ or $yx = \mathcal{F}_0^k(a3) \in \mathcal{F}_0^k(\mathcal{L}_2)$. In the first case, we will have $(va)^{M+2}v \in \mathcal{L}_2$, hence a contradiction. In the second case and since $a \neq 3$, we will obtain $(va3)^{M+2}va \in \mathcal{L}_2$, which is forbidden by (ii).

(ii) Let $\omega \in \{1,2,3\}^*$ and $z \in \{1,2,3\}$ such that $(\omega z)^{M+2}\omega \in \mathcal{F}_0^k(\mathcal{L}_2)$. Let us consider the decomposition of ω in $\omega = x\mathcal{F}_0^k(v)y$. Then, $(\omega z)^{M+2}\omega = x(\mathcal{F}_0^k(v)yzx)^{M+2}\mathcal{F}_0^k(v)y$. Since v does not begin with 3 and since $|yx|_1 + |yx|_2 \leqslant 2$, it follows that there exists a letter $a \in \{1,2,3\}$ such that either $yzx = \mathcal{F}_0^k(a)$ or $yzx = \mathcal{F}_0^k(a3)$. We now have to deal with these two cases.

(a) If $yzx = \mathcal{F}_0^k(a)$, then $(\omega z)^{M+2}\omega = x\mathcal{F}_0^k((va)^{M+2}v)y$. We will obtain (again using the fact that 3 is not a prefix of v) that $(va)^{M+2}v \in \mathcal{L}_1$, hence a contradiction with (ii).

(b) If $yzx = \mathcal{F}_0^k(a3)$, then $a \neq 3$ since $33 \notin \mathcal{L}_2$. If $y \neq \varepsilon$, then $\mathcal{F}_0^k((va3)^{M+2}v)y \in \mathcal{F}_0^k(\mathcal{L}_2)$ and thus it implies that $\mathcal{F}_0^k((va3)^{M+2}v)\mathcal{F}_0^k(a) \in \mathcal{F}_0^k(\mathcal{L}_2)$. It follows that $(va3)^{M+2}va \in \mathcal{L}_2$, which is forbidden by definition of $\mathcal{L}_2$.
If $y = \varepsilon$, then $x = 3^{k+1}$. The fact that $3^{k+1}\mathcal{F}_0^k((va3)^{M+2}v) \in \mathcal{F}_0^k(\mathcal{L}_2)$ thus implies the existence of a letter $b \in \{1,2\}$ such that $\mathcal{F}_0^k(b3)\mathcal{F}_0^k((va3)^{M+2}v) \in \mathcal{F}_0^k(\mathcal{L}_2)$. We thus deduce that $(3va)^{M+2}3v \in \mathcal{L}_2$, hence a contradiction with assertion (ii) of the definition of $\mathcal{L}_2$. □

Lemma 18. *Let k be a positive integer, $k \leqslant M$. Then*

$$\mathcal{G}_0^k(\mathcal{L}_3) \subset \mathcal{L}_1 \cap \mathcal{L}_2.$$

Proof. For the reasons already exposed in the proof of Lemma 16, this proof is similar to the previous one if we exchange the roles played by the letters 1 and 3. □

Lemma 19. *A uniformly recurrent sequence* $\mathbf{u}$ *has a finite index if and only if this is the case for any of its derived sequences. More precisely, for any derived sequence* $\mathbf{v}$ *the following holds*:

- *if* $\mathbf{v}$ *is not asymptotically kth power-free, then* $\mathbf{u}$ *is not asymptotically kth power-free,*
- *if* $\mathbf{v}$ *is asymptotically kth power-free, then* $\mathbf{u}$ *is asymptotically $(k+1)$th power-free.*

Proof. Let $\mathbf{u}$ be a uniformly recurrent sequence defined over the alphabet $\mathcal{A}$, let u be a factor of $\mathbf{u}$ and let $\mathbf{v}$ be the derived sequence of $\mathbf{u}$ on u. Then there exists a morphism φ_u, called the return morphism, such that $\varphi_u(\mathbf{v}) = \mathbf{u}$. It thus follows that if $\mathbf{v}$ is not asymptotically kth power-free, then $\mathbf{u}$ is not asymptotically kth power-free.

Now, let r be the number of return words on u. There thus exist possibly empty but distinct words $v_1, v_2, \ldots, v_r$, defined over $\mathcal{A}$ such that

$$\begin{aligned} \mathcal{B} = \{1, 2, \ldots, r\} &\xrightarrow{\varphi_u} \mathcal{A} \\ i &\longmapsto uv_i. \end{aligned}$$

Let us assume that $\mathbf{u}$ contains arbitrarily long $(k+1)$th powers. Let w be a sufficiently long word such that $w^{k+1} \in \mathcal{L}(\mathbf{u})$. Since w is long enough and $\mathbf{u}$ is uniformly recurrent, the word u has at least two different occurrences in w. The word w can thus be decomposed as

$$w = x\varphi_u(v)y,$$

where x is a strict suffix of the image of a letter by φ_u, $v \in \mathcal{B}$ is not the empty word and where y is a strict prefix of the image of a letter by φ_u. Thus, $w^k = x(\varphi_u(v)yx)^k\varphi_u(v)y$. The fact that the set of return words is a code (see [17]) allows us to say that there is a unique letter $i \in \mathcal{B}$ satisfying $yx = \varphi_u(i)$. It thus follows that $\varphi_u((vi)^k)$ is a factor of $\mathbf{u}$. By definition of the derived sequences, $(vi)^k$ is a factor of $\mathbf{v}$. We thus obtain that if $\mathbf{v}$ is asymptotically kth power-free then $\mathbf{u}$ is asymptotically $(k+1)$th power-free. □

Remark 20. Such a result is of course no more true if we replace arbitrarily long powers by powers.

Proof of Theorem 3. Let $\mathbf{u}$ be a characteristic non-degenerate coding of rotation and let $(a_n, i_n)_{n \geqslant 0}$ be its $\mathcal{D}$-expansion. Then as it is already mentioned in [4], there exists a sequence

$$\mathbf{v} = \lim_{n\to\infty}\left(\prod_{j=0}^{n}\left(\mathcal{F}_{a'_j}^{i'_j} \circ \mathcal{G}_{a'_j}^{1-i'_j}\right)(1)\right)$$

such that:

(i) $\mathbf{v}$ is a derived sequence of $\mathbf{u}$,
(ii) there exists a non-negative integer k, such that $\forall n \geqslant 0$, $a'_n = a_{n+k}$ and $i'_n = i_{n+k}$.

Now, let us assume that for any non-negative integer n:

(i) $a_n \leqslant M$,
(ii) $i_n = i_{n+1} = \cdots = i_{n+M-1} \Rightarrow \exists k, n \leqslant k \leqslant n+M-1$ such that $a_k \neq 0$.

Let us consider the following sets of substitutions:

$$S_1 = \{\mathcal{F}_k, \mathcal{G}_k\colon\ 1 \leqslant k \leqslant M\}, \qquad S_2 = \left\{\mathcal{F}_0^k\colon\ 1 \leqslant k \leqslant M\right\}, \quad \text{and}$$
$$S_3 = \left\{\mathcal{G}_0^k\colon\ 1 \leqslant k \leqslant M\right\}.$$

The condition satisfied by the $\mathcal{D}$-expansion of the sequence **u** ensures the existence of a unique sequence of substitutions $(\sigma_n)_{n\geqslant 0}$, $\sigma_n \in S_1 \cup S_2 \cup S_3$ such that

$$\mathbf{v} = \lim_{n\to\infty} \sigma_0\sigma_1\ldots\sigma_n(1) \tag{4}$$

with the additional conditions

$$\sigma_k \in S_2 \Rightarrow \sigma_{k+1} \notin S_2 \quad \text{and} \quad \sigma_k \in S_3 \Rightarrow \sigma_{k+1} \notin S_3.$$

Then Lemmas 14, 16–18 imply that for any non-negative integer n

$$\sigma_0\sigma_1\ldots\sigma_n(1) \in (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \subset \mathcal{L},$$

which in particular shows that **v** is $(M+3)$th power-free.

Moreover, if there exist infinitely many n such that either $a_n = M$ or $i_n = i_{n+1} = \cdots = i_{n+M-2}$ and $a_k = 0$, $n \leqslant k \leqslant n+M-2$, then at least one of the substitutions $\mathcal{F}_M$, $\mathcal{G}_M$, $\mathcal{F}_0^M$ or $\mathcal{G}_0^M$ appears infinitely often in the composition (4). We thus deduce directly from the definition of these substitutions that in that case **v** contains arbitrarily long $(M+1)$th powers.

The proof thus follows from Lemma 19 and the previous observations. □

6. Words without initial powers: proof of Theorem 8

This section is devoted to the proof of Theorem 8. Our proof is based on an explicit construction of a class of binary codings of rotations without any asymptotic initial power, that is, for which ice is equal to 1. These sequences are obtained by choosing suitable associated $\mathcal{D}$-expansions.

The following definitions will be used in the remainder of this section. Let $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$ be two sequences of positive integers. We define the substitution σ_n over the alphabet $\{1, 2, 3\}$ by

$$\begin{array}{rcl} & \sigma_n & \\ 1 & \longmapsto & 12^{l_n}(13)^{k_n} \\ 2 & \longmapsto & 12^{l_n+1}(13)^{k_n} \\ 3 & \longmapsto & 13. \end{array}$$

It thus follows that $\sigma_n = \mathcal{G}_{l_n} \circ \mathcal{F}_0^{k_n}$, where the substitutions $\mathcal{G}_k$ and $\mathcal{F}_k$ are defined in (1). Let us consider the substitution $\psi_n = \prod_{k=0}^{n-1} \sigma_k$ (we recall that $\prod$ means the composition of morphisms from left to right). Our goal is now to study the initial critical exponent of the sequence $\mathbf{v} = \lim_{n\to\infty} \psi_n(1)$. In what follows, we will also have to consider the sequences $\mathbf{v}_l$ defined by

$$\mathbf{v}_l = \lim_{n\to\infty} \left(\prod_{k=l}^{n-1} \sigma_k\right)(1).$$

Thus, $\psi_l(\mathbf{v}_l) = \mathbf{v}$.

Remark 21. By definition of σ_n, $\psi_n(w) \prec \mathbf{v}$ implies $w \prec \mathbf{v}_n$.

Lemma 22. *Let k be a positive integer and let V_k be the prefix of length k of the sequence $\mathbf{v}$. Let us assume that V_k admits the following decomposition*: $\exists(u,w) \in \{1,2,3\}^*$, $\exists(a,b) \in \{1,2,3\}^2$, $a \neq b$ *such that*

(i) $V_k = uawu$,
(ii) $V_{k+1} = V_k b$.

Then, either $a = 1$ and $b = 2$ or $a = 2$ and $b = 3$.

Proof. The cases ($a = 1$, $b = 3$) and ($a = 3$, $b = 1$) can be suppressed since we easily verify that $11 \notin \mathcal{L}(\mathbf{v})$.

Let us assume that $a = 2$ and $b = 1$. It thus follows that u cannot be the empty word, begins with the letter 1 and admits the word 12^{l_0} as a suffix. This implies the existence of a word u_1 such that $u = \sigma_1(u_1)12^{l_0}$ and of a word w_1 such that $w = 2(13)^{k_0}\sigma_0(w_1)$. Then, $V_k = \sigma_0(u_1 2 w_1 u_1)$ and it derives from 21 that $u_1 2 w_1 u_1$ should be a prefix of $\mathbf{v}_1$ satisfying the assumptions required for V_k and with a smaller length. By iterating this process, we obtain a contradiction.

The case where $a = 3$ and $b = 2$ is similar. □

Lemma 23. *Let k be a positive integer such that V_k satisfies the assumptions required by Lemma* 22. *If moreover $a = 1$ and $b = 2$, then there exists a positive integer n and a word $m \in \{1,2,3\}^*$ such that*:

(i) $3(13)^{k_n-1} \prec m$,
(ii) $u = \psi_n(12^{l_n})\psi_{n-1}(12^{l_{n-1}})\ldots\psi_1(12^{l_1})12^{l_0}$,
(iii) $w = 3(13)^{k_0-1}\psi_1(3(13)^{k_1-1})\ldots\psi_{n-1}(3(13)^{k_{n-1}-1})\psi_n(m)$.

Proof. Since $a = 1$ and $b = 2$, the word 12^{l_0} is a suffix of u and the word $3(13)^{k_0-1}$ is a prefix of w. This implies the existence of a word u_1 such that $u = \sigma_0(u_1)12^{l_0}$ and of a word w_1 such that $w = 3(13)^{k_0-1}\sigma_0(w_1)$. Moreover, we have that $V_k = \sigma_0(u_1 1 w_1 u_1)12^{l_0}$ and thus that $\sigma_0(u_1 1 w_1 u_1)12^{l_0}2 \prec \mathbf{v}$. It thus comes from Remark 21 that $u_1 1 w_1 v_1 2 \prec \mathbf{v}$ and that $|u_1| < |u|$. Then, we can iterate this process until u_n has a minimal length, that is, $u_n = 12^{l_n}$. The result thus follows with $m = w_n$. □

Lemma 24. *Let k be a positive integer and let V_k be the prefix of length k of the sequence $\mathbf{v}$. Let us assume that V_k satisfies the assumptions of Lemma* 22. *If moreover $a = 2$ and $b = 3$, then there exists a positive integer n and a word $m \in \{1,2,3\}^*$ such that*:

(i) $2^{l_n-1} \prec m$,
(ii) $u = \psi_n(1)\psi_{n-1}(1)\ldots\psi_1(1)1$,
(iii) $w = 2^{l_0}(13)^{k_0}\psi_1(2^{l_1}(13)^{k_1})\ldots\psi_{n-1}(2^{l_{n-1}}(13)^{k_{n-1}})\psi_n(m)$.

Proof. Since $a = 2$ and $b = 3$, the letter 1 is a suffix of u and then the word 2^{l_0-1} is a prefix of w. This implies the existence of a word u_1 such that $u = \sigma_0(u_1)1$. If moreover u_1 is not the empty word, that is, if $u \neq 1$, then there exists a word w_1 such that either $w = 2^{l_0-1}(13)^{k_0}\sigma_0(w_1)$ or $w = 2^{l_0}(13)^{k_0}\sigma_0(w_1)$.

In the first case, we will have $V_k = \sigma_0(u_1 1 w_1 u_1)1$, which implies that $V_{k+1} = \sigma_0(u_1 1 w_1 u_1 3)$ since by assumption $V_{k+1} = V_k 3$. Then, by Remark 21 we will have $u_1 1 w_1 u_1 3 \prec \mathbf{v}$. This is in contradiction to the fact that $11 \notin \mathcal{L}(\mathbf{v})$.

We thus have $w = 2^{l_0}(13)^{k_0}\sigma_0(w_1)$ and then $\mathbf{v}_k = \sigma_0(u_1 2 w_1 u_1)1$. Since $V_{k+1} = V_k 3$, we obtain $u_1 2 w_1 u_1 3 \prec \mathbf{v}$ with $|u_1| < |u|$. Then we can iterate this process until u_n has a minimal length, that is, $u_n = 1$. The result thus follows with $m = w_n$. $\square$

Lemma 25. *For any positive integer n, let us consider*

$$\begin{aligned}
u_{1,n} &= \psi_n\big(12^{l_n}\big)\psi_{n-1}\big(12^{l_{n-1}}\big)\ldots\psi_1\big(12^{l_1}\big)12^{l_0},\\
w_{1,n} &= 3(13)^{k_0-1}\psi_1\big(3(13)^{k_1-1}\big)\ldots\psi_n\big(3(13)^{k_n-1}\big),\\
u_{2,n} &= \psi_n(1)\psi_{n-1}(1)\ldots\psi_1(1)1, \quad \textit{and}\\
w_{2,n} &= 2^{l_0}(13)^{k_0}\psi_1\big(2^{l_1}(13)^{k_1}\big)\ldots\psi_n\big(2^{l_n-1}\big).
\end{aligned}$$

Then, $u_{1,n}1w_{1,n}u_{1,n} \prec \mathbf{v}$ and $u_{2,n}2w_{2,n}u_{2,n} \prec \mathbf{v}$.

Proof. An easy induction shows that for any positive integer n,

$$u_{1,n}1w_{1,n} = \psi_{n+1}(1) \quad \text{and} \quad u_{1,n} \prec \psi_{n+1}(2).$$

It follows

$$u_{1,n}1w_{1,n}u_{1,n} \prec \psi_{n+1}(12) \prec \mathbf{v}.$$

Similarly, we obtain by induction that for any positive integer n,

$$u_{2,n}2w_{2,n} = \psi_n\big(12^{l_n}\big) \quad \text{and} \quad u_{2,n} \prec \psi_n(13).$$

It follows

$$u_{2,n}2w_{2,n}v_{2,n} \prec \psi_{n+1}\big(12^{l_n}13\big) \prec \mathbf{v},$$

concluding the proof. $\square$

Lemma 26. *Let us consider the two real sequences $(e_{1,n})_{n\geqslant 0}$ and $(e_{2,n})_{n\geqslant 0}$ respectively defined by*

$$e_{1,n} = 1 + \frac{|u_{1,n}|}{|v_{1,n}1w_{1,n}|} \quad \textit{and} \quad e_{2,n} = 1 + \frac{|u_{2,n}|}{|u_{2,n}2w_{2,n}|}.$$

Then

$$\mathrm{ice}(\mathbf{v}) = \max\Big\{\limsup_{n\to\infty} e_{1,n},\ \limsup_{n\to\infty} e_{2,n}\Big\}.$$

Proof. It first follows from Lemma 25 that

$$\mathrm{ice}(\mathbf{v}) \geqslant \max\Big\{\limsup_{n\to\infty} e_{1,n},\ \limsup_{n\to\infty} e_{2,n}\Big\}.$$

Now, let us consider an initial power of $\mathbf{v}$. Since $\mathbf{v}$ clearly does not begin with any square, such an initial power can be decomposed as $vu \prec \mathbf{v}$, where u is a strict prefix of the word v. We can assume without any restriction that vu is a maximal initial power. That is, there exists a letter b such that $vub \prec \mathbf{v}$ and ub is not a prefix of v. Since u is a strict prefix of the word v and ub is not a prefix of v, we can decomposed the word v as uaw, a being a letter different to the letter b. Then it follows from Lemma 22 that either $a = 1$ and $b = 2$ or $a = 2$ and $b = 3$. By Lemmas 23 and 24, we have

$$\mathrm{ice}(\mathbf{v}) \leqslant \max\Big\{\limsup_{n\to\infty} e_{1,n},\ \limsup_{n\to\infty} e_{2,n}\Big\},$$

hence the proof. □

Proof of Theorem 8. Let $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$ be two increasing sequences of positive integers satisfying $l_n = o(k_n)$. Let us note that there exists a non-countable numbers of such sequences. All the other quantities are defined as previously in this section.

Now let us consider the sequence $(a_n, i_n)_{n\geqslant 0}$ defined by

$$(l_0, 0), \underbrace{(0,1), (0,1), \ldots, (0,1)}_{k_0 \text{ times}}, (l_1, 0), \underbrace{(0,1), (0,1), \ldots, (0,1)}_{k_1 \text{ times}}, \ldots,$$
$$, \ldots, l_n, 0), \underbrace{(0,1), (0,1), \ldots, (0,1)}_{k_n \text{ times}}, \ldots.$$

By Theorem 12, we know that there exists a characteristic coding of rotation $\mathbf{u}$ defined by $\mathbf{u} = \phi_1(\mathbf{v})$, where

$$\mathbf{v} = \lim_{n\to\infty}\left(\prod_{j=0}^{n}\big(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\big)(1)\right)$$

and where ϕ_1 is defined by

$$\begin{aligned} 1 &\longmapsto 1 \\ 2 &\longmapsto 122 \\ 3 &\longmapsto 12. \end{aligned}$$

The definition of ϕ_1 implies that if $\mathrm{ice}(\mathbf{v}) = 1$ then $\mathrm{ice}(\mathbf{u}) = 1$. The last step to prove Theorem 8 is thus to show that $\mathrm{ice}(\mathbf{v}) = 1$ for our choice of sequences $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$. In order to prove this, we first have to remark that our sequence $\mathbf{v}$ corresponds to the sequence $\mathbf{v}$ defined at the beginning of this section with our choice of sequence $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$. We thus can keep the previous notation used in this section.

We now have to recall that for any positive integer n,

$$\left|\psi_n(13)\right| \geqslant \left|\psi_n(2)\right| \geqslant \left|\psi_n(1)\right|.$$

Since $l_n|\psi_n(2)| \leqslant |\psi_n(12^{l_n})| \leqslant (l_n+1)|\psi_n(2)|$ and $|\psi_n(3(13)^{k_n-1})| \geqslant (k_n-1)|\psi_n(2)|$, it follows that

$$\limsup_{n\to\infty}(e_{1,n}) \leqslant 1 + \sum_{j=0}^{n}(l_j+1) \Big/ \sum_{j=0}^{n}(l_j+k_j-1).$$

Since $|\psi_n(2^{l_n}(13)^{k_n})| \geqslant |\psi_n(2^{l_n-1})| \geqslant (l_n-1)|\psi_n(2)|$, it follows that

$$\limsup_{n\to\infty}\,(e_{2,n}) \leqslant 1 + n \Big/ \sum_{j=0}^{n}(l_j-1).$$

Since the sequences $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$ are increasing and satisfy $l_n = o(k_n)$, we obtain that the real sequences $(e_{1,n})_{n\geqslant 0}$ and $(e_{2,n})_{n\geqslant 0}$ together vanish when n increases. By Lemma 26, we thus have $\mathrm{ice}(\mathbf{v}) = 1$, concluding the proof. $\square$

Remark 27. If the sequences $(l_n)_{n\geqslant 0}$ and $(k_n)_{n\geqslant 0}$ are constant ones (respectively equal to l and to k), we obtain following [2] a binary coding of rotation with parameters lying in a same quadratic field. In that case, the parameters can be computed explicitly and necessarily the initial critical exponent is greater than 1. However, if l is large enough and if k is large enough with respect to l, we can construct explicit codings with an initial critical exponent less than $3/2$ and thus for which Theorem 5 could not be applied. For instance, this is the case of the binary coding of rotation associated with parameters

$$\left(\alpha = \frac{785-\sqrt{25277}}{1882},\ \beta = \frac{1037}{1882} + \frac{169\sqrt{25277}}{295474}\right),$$

which has the periodic $\mathcal{D}$-expansion $[\,\overline{(12,0)(0,1)^{145}}\,]$.

7. Segment expansion factor and the transcendence of some continued fractions: proof of Theorem 10

In the previous section, we have shown the existence of many binary codings of rotations without any asymptotic initial power. This difference with respect to the Sturmian case has a particular importance for proving the transcendence of associated continued fractions.

Indeed, we know now that we cannot apply Theorem 5 to this class of sequences. However, we will prove Theorem 10 by using a recent result due to J.L. Davison [16] which generalizes Theorem 5. Roughly speaking, this result allows to replace asymptotic initial powers by asymptotic almost initial powers in the method given by [5].

We first have to precise what is the meaning of almost initial powers by recalling the following definition introduced in [16].

Definition 28. Suppose $\gamma \in \mathbb{R}$, $\gamma \geqslant 1$. The infinite word $\mathbf{u} = u_0u_1u_2\ldots$ is said to have a segment expansion factor $\geqslant \gamma$ if it begins, for every positive integer n, in $U_nV_nW_n$, where:

- U_n is a possibly empty word,
- $\lim_{n\to\infty} |V_n| = \infty$,
- $W_n \prec V_n^s$ for some positive integer s,
- $\frac{|U_nV_nW_n|}{|U_n|+|U_nV_n|} \geqslant \gamma$.

Before stating the generalization of Theorem 5 obtained by J.L. Davison, we introduce a quantity which naturally takes place in the study of continued fractions.

Definition 29. Let α be a real number and let $(p_n/q_n)_{n\geqslant 0}$ be the sequence of convergents associated with α. Then, we define the quantity $L(\alpha)$ by

$$L(\alpha) = \limsup_{n\to\infty} \frac{1}{n}\log q_n \Big/ \liminf_{n\to\infty} \frac{1}{n}\log q_n.$$

The real number α has a Lévy constant when $\lim_{n\to\infty} \frac{1}{n}\log q_n$ exists, that is, when $L(\alpha) = 1$.

We are now ready to state the main result of [16].

Theorem 30 (Davison [16]). *Let α be a real number having the sequence $\mathbf{u} = u_0u_1u_2\ldots$ as continued fraction expansion. Let us assume that the sequence $\mathbf{u}$ has segment expansion factor $\geqslant \gamma$. Then, if $\gamma/L(\alpha) > 3/2$, the number α is transcendental.*

The remainder of this section is devoted to auxiliary results needed for proving Theorem 10 and to the proof itself.

Lemma 31. *Let α be a real number having the sequence $\mathbf{u} = u_0u_1u_2\ldots$ as continued fraction expansion. Let $\overline{\mathcal{O}(\mathbf{u})}$ be the closure of the orbit under the shift S of the sequence $\mathbf{u}$. If the dynamical system $(\overline{\mathcal{O}(\mathbf{u})}, S)$ is uniquely ergodic, then α has a Lévy constant.*

Proof. It suffices to follow the proof of Theorem 10 at the end of [5]. In fact, the authors of [5] prove the weaker following result: α has a Lévy constant if the associated continued fraction is a fixed point of a primitive substitution. But what is only used in their proof is the fact that the subshift associated with such a sequence is uniquely ergodic. $\square$

Lemma 32. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two infinite words and* ϕ *be a morphism such that* $\phi(\mathbf{v}) = \mathbf{u}$. *If every letter in* $\mathbf{v}$ *has a frequency and if* $\mathbf{v}$ *has a segment expansion factor* $> 3/2$, *then* $\mathbf{u}$ *has a segment expansion factor* $> 3/2$.

Proof. Let $\mathbf{v}$ be an infinite sequence defined over the alphabet $\mathcal{A} = \{1, 2, \dots, d\}$ whose every letter has a frequency and with a segment expansion factor $> 3/2$, let $\mathbf{u}$ be an infinite sequence defined over the alphabet $\mathcal{B} = \{1, 2, \dots, r\}$ and let ϕ be a morphism from $\mathcal{A}$ to $\mathcal{B}$ such that $\phi(\mathbf{v}) = \mathbf{u}$. Since $\mathbf{v}$ has a segment expansion factor $> 3/2$, there exist $\varepsilon > 0$ and three sequences of finite words $(U_n)_{n\geqslant 0}$, $(V_n)_{n\geqslant 0}$ and $(W_n)_{n\geqslant 0}$ such that:

- $\lim_{n\to\infty} |V_n| = \infty$,
- $W_n \prec V_n^s$ for some positive integer s,
- $\frac{|U_n V_n W_n|}{|U_n|+|U_n V_n|} \geqslant \frac{3}{2} + \varepsilon$.

Let us denote by $f(i)$ the frequency of the letter i in the sequence $\mathbf{v}$. We thus obtain that

$$\bigl|\phi(U_n V_n W_n)\bigr| = \sum_{i=1}^{d} f(i)\bigl|\phi(i)\bigr||U_n V_n W_n| + o\bigl(\bigl|\phi(U_n V_n W_n)\bigr|\bigr) \quad \text{and}$$

$$\bigl|\phi(U_n U_n V_n)\bigr| = \sum_{i=1}^{d} f(i)\bigl|\phi(i)\bigr||U_n U_n V_n| + o\bigl(\bigl|\phi(U_n U_n V_n)\bigr|\bigr).$$

This implies

$$\begin{aligned}\frac{|\phi(U_n V_n W_n)|}{2|\phi(U_n)| + |\phi(V_n)|} &= \frac{|\phi(U_n V_n W_n)|}{|\phi(U_n U_n V_n)|} = \frac{|\phi(U_n V_n W_n)|}{|U_n U_n V_n|} \times \frac{|U_n U_n V_n|}{|\phi(U_n U_n V_n)|} \\ &= \frac{|U_n V_n W_n|}{2|U_n| + |V_n|} + o(1) \geqslant \frac{3}{2} + \varepsilon + o(1),\end{aligned}$$

which ends the proof. □

Definition 33. Let $\mathcal{S} = \{\mathcal{F}_k\colon k \geqslant 0\} \cup \{\mathcal{G}_k\colon k \geqslant 0\}$, with the substitutions $\mathcal{F}_k$ and $\mathcal{G}_k$ being defined in (1) and let $\mathcal{S}^*$ be the free monoid (for the composition of morphisms) generated by $\mathcal{S}$. Let U and W be two finite words on $\{1, 2, 3\}$ and let ε be a positive number. Then, $U \ll_\varepsilon W$ if for any element ϕ in $\mathcal{S}^*$ we have:

$$\bigl|\phi(W)\bigr| \geqslant (1+\varepsilon)\bigl|\phi(U)\bigr|.$$

Lemma 34. *Let* U *and* W *be two finite words on* $\{1, 2, 3\}$ *and let* ε *be a positive number. Let us assume that either*

$$\bigl(|W|_2 - |U|_2\bigr) - \bigl(\max\bigl(|U|_1 - |W|_1, 0\bigr) + \max\bigl(|U|_3 - |W|_3, 0\bigr)\bigr) \geqslant \varepsilon |U| \tag{5}$$

or

$$\min\bigl(|W|_1 - |U|_1, |W|_3 - |U|_3\bigr) - \max\bigl(|U|_2 - |W|_2, 0\bigr) \geqslant \varepsilon |U|, \tag{6}$$

then $U \ll_\varepsilon W$.

Proof. Let ϕ be an element of $\mathcal{S}^*$. Then, ϕ is the composition of a finite number n of substitutions of type $\mathcal{F}_k$ and $\mathcal{G}_k$. We first prove by induction on n the following inequalities:

$$\bigl|\phi(13)\bigr| > \bigl|\phi(2)\bigr| \geqslant \max\bigl\{\bigl|\phi(1)\bigr|, \bigl|\phi(2)\bigr|, \bigl|\phi(3)\bigr|\bigr\}. \tag{7}$$

If $n = 0$ then ϕ is the identity and thus the result is obvious. Now, let us assume that (7) is satisfied for a given l and let ψ be an element of $\mathcal{S}^*$ obtained as a product of $l + 1$ substitutions of type $\mathcal{F}_k$ and $\mathcal{G}_k$. Then there exists an element ξ of $\mathcal{S}^*$ obtained as a product of l substitutions of type $\mathcal{F}_k$ and $\mathcal{G}_k$ and an integer r such that $\psi = \xi \circ \mathcal{F}_r$ or $\psi = \xi \circ \mathcal{G}_r$. By assumption, we have:

$$\bigl|\xi(13)\bigr| > \bigl|\xi(2)\bigr| \geqslant \max\bigl\{\bigl|\xi(1)\bigr|, \bigl|\xi(2)\bigr|, \bigl|\xi(3)\bigr|\bigr\}. \tag{8}$$

By definition of the substitutions $\mathcal{F}_r$, we have:

$$\bigl|\psi(13)\bigr| = \bigl|\xi(13)\bigr| + r\bigl|\xi(2)\bigr| + \bigl|\xi(3)\bigr|, \qquad \bigl|\psi(2)\bigr| = \bigl|\xi(2)\bigr| + r\bigl|\xi(2)\bigr| + \bigl|\xi(3)\bigr|,$$
$$\bigl|\psi(3)\bigr| = r\bigl|\xi(2)\bigr| + \bigl|\xi(3)\bigr|, \qquad \bigl|\psi(1)\bigr| = \bigl|\xi(1)\bigr| + \bigl|\xi(3)\bigr|.$$

Then, (8) implies that

$$\bigl|\psi(13)\bigr| > \bigl|\psi(2)\bigr| \geqslant \max\bigl\{\bigl|\psi(1)\bigr|, \bigl|\psi(2)\bigr|, \bigl|\psi(3)\bigr|\bigr\}.$$

By symmetry, the above inequalities are also true if we replace $\mathcal{F}_r$ by $\mathcal{G}_r$ and if we exchange the role played by the letters 1 and 3.

Now, let W and U be two finite words on $\{1, 2, 3\}$ satisfying (5) for a given positive ε and let ϕ be an element of $\mathcal{S}^*$. It thus follows that

$$\begin{aligned}\bigl|\phi(W)\bigr| - \bigl|\phi(U)\bigr| &= \sum_{i=1}^{3}\bigl(|W|_i - |U|_i\bigr)\bigl|\phi(i)\bigr| \\ &\geqslant \bigl(|W|_2 - |U|_2\bigr)\bigl|\phi(2)\bigr| - \sum_{i\in\{1,3\}}\bigl(\max\bigl\{|U|_i - |W|_i, 0\bigr\}\bigl|\phi(i)\bigr|\bigr).\end{aligned}$$

Following (7), we have:

$$\begin{aligned}\bigl|\phi(W)\bigr| - \bigl|\phi(U)\bigr| &\geqslant \bigl(|W|_2 - |U|_2\bigr)\bigl|\phi(2)\bigr| - \Bigl(\sum_{i\in\{1,3\}}\max\bigl\{|U|_i - |W|_i,\ 0\bigr\}\Bigr)\bigl|\phi(2)\bigr| \\ &\geqslant \Bigl(\bigl(|W|_2 - |U|_2\bigr) - \sum_{i\in\{1,3\}}\max\bigl\{|U|_i - |W|_i,\ 0\bigr\}\Bigr)\bigl|\phi(2)\bigr|.\end{aligned}$$

Then (5) and (7) imply that

$$\left|\phi(W)\right| - \left|\phi(U)\right| \geqslant \varepsilon|U|\left|\phi(2)\right| \geqslant \varepsilon\left|\phi(U)\right|.$$

The case where W and U satisfy (6) is similar, hence the proof. □

We will use in the remainder of this section a notation similar to the one introduced in Section 6. Let us consider a sequence $(a_n, i_n)_{n\geqslant 0} \in (\mathbb{N} \times \{0, 1\})^{\mathbb{N}}$, with $(a_n)_{n\geqslant 0}$ not eventually vanishing and with $(i_n)_{n\geqslant 0}$ not eventually constant. Let $\mathbf{v}$ be the infinite sequence defined by:

$$\mathbf{v} = \lim_{n\to\infty}\left(\prod_{j=0}^{n}\left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)(1)\right), \tag{9}$$

where the substitutions $\mathcal{F}_k$ and $\mathcal{G}_k$ are defined in (1). Then, if k is a positive integer we denote by $\mathbf{v}_k$ the sequence

$$\lim_{n\to\infty}\left(\prod_{j=k}^{n}\left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)(1)\right)$$

and by ϕ_k the morphism

$$\prod_{j=0}^{k-1}\left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)$$

such that $\phi_k(\mathbf{v}_k) = \mathbf{v}$.

Lemma 35. *Let ε be a positive number. If there exist a pair of sequences $(W_l, U_l)_{l>0}$ of finite words on $\{1, 2, 3\}$ with $U_l \ll_\varepsilon W_l$ and an increasing sequence of integers $(k_l)_{l>0}$, such that for any positive integer l the sequence $\mathbf{v}_{k_l}$ begins in $W_l U_l W_l$, then $\mathbf{u}$ has a segment expansion factor $> 3/2$.*

Proof. If $V \ll_\varepsilon W$ we obtain by (7) and by Definition 33 that

$$\left|\phi_k(W)\right| \geqslant (1+\varepsilon)\left|\phi_k(V)\right|,$$

for every positive integer k. By assumptions, the sequence $\mathbf{u}$ begins in $\phi_{k_l}(W_l U_l W_l)$ for every positive integer l. It thus follows that $\mathbf{u}$ has a segment expansion factor

$$\geqslant \liminf_{l\to\infty} \frac{|\phi_{k_l}(W_l U_l W_l)|}{|\phi_{k_l}(W_l U_l)|}.$$

But (7) implies that

$$\liminf_{l\to\infty}\frac{|\phi_{k_l}(W_lU_lW_l)|}{|\phi_{k_l}(W_lU_l)|}\geqslant 1+\liminf_{l\to\infty}\frac{|\phi_{k_l}(W_l)|}{|\phi_{k_l}(W_l)|(1+\frac{1}{1+\varepsilon})}=\frac{3}{2}+\frac{\varepsilon}{2(2+\varepsilon)}>\frac{3}{2},$$

concluding the proof. □

Lemma 36. *Let ε be a positive number. If there exist a pair of sequences $(W_l, U_l)_{l>0}$ of finite words on $\{1,2,3\}$ and an increasing sequence of integers $(k_l)_{l>0}$, such that for all positive integer l:*

- *the sequence $\mathbf{v}_{k_l}$ begins in $1W_lU_l2W_l$,*
- *$U_l2 \ll_\varepsilon W_l$,*

then $\mathbf{u}$ *has a segment expansion factor $> 3/2$.*

Proof. Let $\varepsilon > 0$ and let (W, U) be a pair of words on $\{1,2,3\}$ with $U \ll_\varepsilon W$. Let us first consider the image of the word $1WU2W$ by the two substitutions $\mathcal{F}_k$ and $\mathcal{G}_k$. We have $\mathcal{F}_k(1WU2W) = 13\mathcal{F}_k(W)\mathcal{F}_k(U)2^k3\mathcal{F}_k(W) = 1W'U'2W$, with $W' = 3\mathcal{F}_k(W)$ and $U' = \mathcal{F}_k(U)2^k \prec \mathcal{F}_k(U2)$. If we apply $\mathcal{G}_k$, we obtain $\mathcal{G}_k(1WU2W) = 12^k\mathcal{G}_k(W)\mathcal{G}_k(U)12^{k+1}\mathcal{G}_k(W) = 1W'U'2W$, with $W' = 2^k\mathcal{G}_k(W)$ and $U' = \mathcal{G}_k(U)1 \prec \mathcal{G}_k(U2)$. In both cases, we have $U' \ll_\varepsilon W'$ and $|W'| > |W|$.

Then, an easy induction shows that one can found a pair of sequences $(W'_l, U'_l)_{l>0}$ of finite words on $\{1,2,3\}$ with $U'_l \ll_\varepsilon W'_l$ and $\lim_{l\to\infty}|W'_l| = \infty$ and such that $\phi_{k_l}(1W_lU_l2W_l) = 1W'_lU'_l2W'l$. It thus follows from our assumption that the sequence $\mathbf{u}$ begins for every positive integer l in $1W'_lU'_l2W'_l$. The sequence $\mathbf{u}$ thus has a segment expansion factor

$$\geqslant \liminf_{l\to\infty}\frac{|1W'_lU'_l2W'l|}{|1|+|1W'_lU'_l2|}=\liminf_{l\to\infty}\frac{|W'_lU'_lW'l|}{|W'_lU'_l|}.$$

Since $U'_l \ll_\varepsilon W'_l$, it follows that

$$\liminf_{l\to\infty}\frac{|W'_lU'_lW'l|}{|W'_lU'_l|}\geqslant 1+\liminf_{l\to\infty}\frac{|W'l|}{|W'_l|(1+\frac{1}{1+\varepsilon})}=\frac{3}{2}+\frac{\varepsilon}{2(2+\varepsilon)}>\frac{3}{2},$$

concluding the proof. □

Lemma 37. *The sequence* $\mathbf{v}$ (*defined in* (9)) *has a segment expansion factor $\geqslant 3/2$.*

Proof. The assumption on the sequence $(a_n, i_n)_{n\geqslant 0}$ allows us to claim that at least one of the following holds:

(a) $(0,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$,

(b) there exist two sequences of positive integers $(j_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ such that for every n, the block $(j_n, 0)(k_n, 0)$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(c) there exists a sequence of positive integers $(j_n)_{n\in\mathbb{N}}$, $j_n \geqslant 4$, such that for every n, the block $(j_n, 0)$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(d) there exists a sequence of positive integers $(j_n)_{n\in\mathbb{N}}$, $j_n \geqslant 5$, such that for every n, the block $(j_n, 1)$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(e) there exists a sequence of positive integers $(j_n)_{n\in\mathbb{N}}$, $j_n \geqslant 14$, such that for every n, the block $(0, 1)^{j_n}$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(f) there exist two sequences of positive integers $(j_n)_{n\in\mathbb{N}}$, $1 \leqslant j_n \leqslant 3$, and $(k_n)_{n\in\mathbb{N}}$, $1 \leqslant k_n \leqslant 4$, such that for every n, the block $(j_n, 0)(k_n, 1)$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(g) there exist three increasing sequences of integers $(j_n)_{n\geqslant 0}$, $1 \leqslant j_n \leqslant 3$, $(k_n)_{n\in\mathbb{N}}$, $1 \leqslant k_n \leqslant 13$, and $(l_n)_{n\in\mathbb{N}}$, $1 \leqslant l_n \leqslant 4$, such that for every n, the block $(j_m, 0)(0, 1)^{k_n}(l_n, 1)$ appears in $(a_n, i_n)_{n\geqslant 0}$,

(h) there exist three increasing sequences of integers $(j_n)_{n\geqslant 0}$, $j_n \leqslant 3$, $(k_n)_{n\in\mathbb{N}}$, $k_n \leqslant 13$, and $(l_n)_{n\in\mathbb{N}}$, $l_n \leqslant 4$, such that for every n, the block $(j_m, 0)(0, 1)^{k_n}(l_n, 0)$ appears in $(a_n, i_n)_{n\geqslant 0}$.

(a) In this first case, we obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_0(\mathbf{v}_{k+1})$, which implies that $\mathbf{v}_k$ begins in the word 11 since

$$\begin{array}{rcl} & \mathcal{G}_0 & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 12 \\ 3 & \longmapsto & 13. \end{array}$$

It easily follows that the sequence $\mathbf{v}$ begins in arbitrarily long squares and thus that it has a segment expansion factor $\geqslant 2$.

Now we can assume without restriction that $(0, 0)$ does not appear in $(a_n, i_n)_{n\geqslant 0}$. This easily implies that for any non-negative integer k, the sequence $\mathbf{v}_k$ does not begin in the word 11.

(b) In this case, we obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_{j_n} \circ \mathcal{G}_{k_n}(\mathbf{v}_{k+2})$, which implies that $\mathbf{v}_k$ begins in the word $12^{k_n}12^{k_n}$ since

$$\begin{array}{rcl} & \mathcal{G}_{j_n} \circ \mathcal{G}_{k_n} & \\ 1 & \longmapsto & 12^{j_n}(12^{j_n+1})^{k_n} \\ 2 & \longmapsto & 12^{j_n}(12^{j_n+1})^{k_n+1} \\ 3 & \longmapsto & 12^{j_n}13. \end{array}$$

It follows that the sequence $\mathbf{v}$ begins in arbitrarily long squares and thus that it has a segment expansion factor $\geqslant 2$.

Now we can assume without any loss of generality that there is no block of the form $(m, 0)(k, 0)$ in the sequence $(a_n, i_n)_{n\geqslant 0}$. This means that if for positive integers k and n, $\mathbf{v}_k = \mathcal{G}_n(\mathbf{v}_{k+1})$ then there exists a non-negative integer m such that $\mathbf{v}_{k-1} = \mathcal{F}_m(\mathbf{v}_k)$.

(c) In this case, we obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_{j_n}(\mathbf{v}_{k+1})$, which implies that $\mathbf{v}_k$ begins in the word 12^4 since $\mathcal{G}_{j_n}(1) = 12^{j_n}$ and $j_n \geqslant 4$. It follows that the sequence $\mathbf{v}$ begins in $\phi_k(12^4)$ and that it thus has a segment expansion factor

$$\geqslant \liminf_{k\geqslant 0}\left\{\frac{|\phi_k(12^4)|}{2|\phi_k(1)| + |\phi_k(2)|}\right\} \geqslant 1 + \frac{2}{3} > \frac{3}{2},$$

since following (7) we have $|\phi_k(2)| \geqslant |\phi_k(1)|$.

(d) We obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{F}_{j_n}(\mathbf{v}_{k+1})$. Since $\mathbf{v}_{k+1}$ does not begin in 11, it should begin either in 12 or in 13. This implies that $\mathbf{v}_k$ begins in the word 132^5 since $\mathcal{F}_{j_n}(12) = 132^{j_n+1}3$, $\mathcal{F}_{j_n}(13) = 132^{j_n}3$ and $j_n \geqslant 5$. It follows that the sequence $\mathbf{v}$ begins in $\phi_k(132^5)$ and that it thus has a segment expansion factor

$$\geqslant \liminf_{k\geqslant 0}\left\{\frac{|\phi_k(132^5)|}{2|\phi_k(13)| + |\phi_k(2)|}\right\} \geqslant 1 + \frac{2}{3} > \frac{3}{2},$$

since following (7) we have $|\phi_k(2)| \geqslant \max\{|\phi_k(1)|, |\phi_k(3)|\}$.

Now, we can assume without restriction that for every n, a_n is at most equal to 4.

(e) We obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{F}_0^{j_n}(\mathbf{v}_{k+j_n})$, which implies that $\mathbf{v}_k$ begins in the word 13^{14} since $j_n \geqslant 14$ and since

$$\begin{array}{rcl} & \mathcal{F}_0^{j_n} & \\ 1 & \longmapsto & 13^{j_n} \\ 2 & \longmapsto & 23^{j_n} \\ 3 & \longmapsto & 3. \end{array}$$

Moreover, we can assume without any loss of generality that j_n is chosen such that $\mathbf{v}_{k-1} \neq \mathcal{F}_0(\mathbf{v}_k)$. It follows that the sequence $\mathbf{v}_{k-1}$ begins either in $\mathcal{F}_m(13^{14})$ or in $\mathcal{G}_m(13^{14})$, $1 \leqslant m \leqslant 5$. The sequence $\mathbf{v}$ thus has a segment expansion factor

$$\begin{aligned} &\geqslant \liminf_{k\geqslant 1,\, 1\leqslant m\leqslant 5}\left\{\min\left(\frac{|\phi_{k-1}(13(2^m3)^{14})|}{2|\phi_{k-1}(13)| + |\phi_{k-1}(2^m3)|}, \frac{|\phi_{k-1}(12^m(13)^{14})|}{2|\phi_{k-1}(12^m)| + |\phi_{k-1}(13)|}\right)\right\} \\ &\geqslant \liminf_{k\geqslant 1}\left\{\frac{|\phi_{k-1}(12^5(13)^{14})|}{2|\phi_{k-1}(12^5)| + |\phi_{k-1}(13)|}\right\} \geqslant 1 + \frac{7}{13} > \frac{3}{2}, \end{aligned}$$

since following (7) we have $|\phi_{k-1}(13)| > |\phi_{k-1}(2)| \geqslant \max\{|\phi_{k-1}(1)|, |\phi_{k-1}(3)|\}$.

(f) We obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_{j_n} \circ \mathcal{F}_{k_n}(\mathbf{v}_{k+2})$. Since $\mathbf{v}_{k+2}$ does not begin in 11 and since $k_n \geqslant 1$, the sequence $\mathbf{v}_{k+1}$ should begin in 132 and the sequence $\mathbf{v}_k$ thus begin in $12^{j_n}1312^{j_n}$.

If there are infinitely many n for which $j_n > 1$, then using (5) of Lemma 34 we obtain $13 \ll_{1/2} 12^{j_n}$. Lemma 35 thus implies that $\mathbf{v}$ has a segment expansion factor $> 3/2$.

Else, we can assume without restriction that j_n is always equal to 0. We thus found an infinite number of k such that $\mathbf{v}_k$ begins in the word 121312. Moreover, we know that $\mathbf{v}_{k-1} = \mathcal{F}_m(\mathbf{v}_k)$, $0 \leqslant m \leqslant 4$, since as we have already assumed that a block of the form $(m,0),(n,0)$ cannot appear in $(a_n,i_n)_{n\geqslant 0}$. We thus have that $\mathbf{v}_{k-1}$ begins in $132^{m+1}3132^m3132^{m+1}3$. We can rewrite this word as $W_nU_nW_n$, where $W_n = 132^{m+1}3$ and $U_n = 132^m3$. In view of Lemma 36, it remains to prove the existence of a positive ε such that for every n, $U_n \ll_\varepsilon W_n$. Following (5) of Lemma 34, we obtain that

$$132^m3 \ll_{\frac{1}{m+3}} 132^{m+1}3.$$

Since $m \leqslant 4$, we always have $132^m3 \ll_{1/7} 132^{m+1}3$. Lemma 35 thus implies that $\mathbf{v}$ has a segment expansion factor $> 3/2$.

(g) We obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_{j_n} \circ \mathcal{F}_0^{k_n} \circ \mathcal{F}_{l_n}(\mathbf{v}_{k+k_n+2})$. Since $\mathbf{v}_{k+k_n+2}$ does not begin in 11 and since $l_n \geqslant 1$, the sequence $\mathbf{v}_{k+k_n+1}$ should begin in 132. It thus follows that the sequence $\mathbf{v}_k$ begins in the word $12^{j_n}(13)^{k_n+1}12^{j_n+1}(13)^{k_n}1$ since we recall that

$$\begin{array}{ccc} & \mathcal{G}_{j_n} \circ \mathcal{F}_0^{k_n} & \\ 1 & \longmapsto & 12^{j_n}(13)^{k_n} \\ 2 & \longmapsto & 12^{j_n}(13)^{k_n} \\ 3 & \longmapsto & 13. \end{array} \tag{10}$$

We first have to note that $\mathbf{v}_{k-1} = \mathcal{F}_m(\mathbf{v}_k)$, $0 \leqslant m \leqslant 4$, since as we have already assumed that a block of the form $(m,0),(j_n,0)$ cannot appear in $(a_n,i_n)_{n\geqslant 0}$. We thus have that $\mathbf{v}_{k-1}$ begins in $13(2^{m+1}3)^{j_n}(132^m3)^{k_n+1}13(2^{m+1}3)^{j_n+1}(132^m3)^{k_n}13$. We can rewrite this word as $1W_nU_n2W_n$, where $W_n = 3(2^{m+1}3)^{j_n}(132^m3)^{k_n}13$ and $U_n = 2^m3132^m$. In view of Lemma 36, it remains to prove the existence of a positive ε such that for every n, $U_n2 \ll_\varepsilon W_n$. Following (6) of Lemma 34, we obtain that

$$2^m3132^{m+1} \ll_{\frac{1}{2m+4}} 3\big(2^{m+1}3\big)^{j_n}\big(132^m3\big)^{k_n}13.$$

Since $m \leqslant 4$, we always have $132^m3 \ll_{1/12} 132^{m+1}3$. Lemma 35 thus implies that $\mathbf{v}$ has a segment expansion factor $> 3/2$.

(h) We obtain that for an infinite number of k, $\mathbf{v}_k = \mathcal{G}_{j_n} \circ \mathcal{F}_0^{k_n} \circ \mathcal{G}_{l_n}(\mathbf{v}_{k+k_n+2})$. Since $l_n \geqslant 1$, the sequence $\mathbf{v}_{k+k_n+1}$ begins in 12. It thus follows in view of (10) that the sequence $\mathbf{v}_k$ begins in $12^{j_n}(13)^{k_n}12^{j_n+1}(13)^{k_n}1$. We can rewrite this word as $1W_nU_n2W_n$, where $W_n = 2^{j_n}(13)^{k_n}1$ and U_n is the empty word. In view of Lemma 36, it remains to prove the existence of a positive ε such that for every n, $U_n2 = 2 \ll_\varepsilon W_n$. Since j_n and k_n are positive, we can use (6) of Lemma 34 to obtain that for every n we have $2 \ll_1 2^{j_n}(13)^{k_n}1$. Lemma 35 thus implies that $\mathbf{v}$ has a segment expansion factor $> 3/2$, concluding the proof. □

Proof of Theorem 10. If $\mathbf{u}$ denotes the natural characteristic coding of an i.d.o.c. three-interval exchange, then it is proved in [2] that there exist a non-erasing morphism ϕ defined on $\{1, 2, 3\}$ and a sequence $(a_n, i_n)_{n\geqslant 0} \in (\mathbb{N} \times \{0, 1\})^{\mathbb{N}}$, with $(a_n)_{n\geqslant 0}$ not eventually vanishing and with $(i_n)_{n\geqslant 0}$ not eventually constant, such that $\mathbf{u} = \phi(\mathbf{v})$, where the sequence $\mathbf{v}$ is the natural coding of another i.d.o.c. three-interval exchange defined by:

$$\mathbf{v} = \lim_{n\to\infty}\left(\prod_{j=0}^{n}\left(\mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j}\right)(1)\right).$$

By Lemma 37, we obtain that $\mathbf{v}$ has a segment expansion factor $> 3/2$. Since it is well known that any letter in the natural coding of an i.d.o.c. three-interval exchange admits a frequency, then Lemma 32 implies that $\mathbf{u}$ also has a segment expansion factor $> 3/2$. Moreover, it is also well known that the subshift associated with the natural coding of an i.d.o.c. three-interval exchange is uniquely ergodic (see, for instance, [36]). It thus follows from Lemma 31 that sequence $\mathbf{u}$ satisfies the condition required in Theorem 30, concluding the proof in this case.

If $\mathbf{u}$ denotes the natural coding of a non-periodic three-interval exchange which does not satisfy the i.d.o.c., then it is shown in [2] that $\mathbf{u}$ must be quasi-Sturmian and thus the result is already proved in Theorem 6.

If $\mathbf{u}$ denotes a nondegenerate characteristic coding of rotation, then by Theorem 11 we know that there exist a characteristic coding of an i.d.o.c. three-interval exchange $\mathbf{v}$ and a non-erasing morphism ϕ from $\{1, 2, 3\}$ into $\{1, 2\}$ such that either $\mathbf{u} = \phi(\mathbf{v})$ or $\mathbf{u} = 1S(\phi(\mathbf{v}))$, where S denotes the classical shift transformation. In these two cases, we easily obtain that the sequence $\mathbf{u}$ has a segment expansion factor $> 3/2$. Indeed, since we have already noticed that any letter in the natural coding of an i.d.o.c. three-interval exchange admits a frequency, we can apply Lemma 32. The subshift associated with an irrational binary coding of rotation being uniquely ergodic (see, for instance, [37]), Lemma 31 implies that the continued fraction associated with such a sequence satisfies the conditions of Theorem 30, concluding the proof in this case.

Finally, if $\mathbf{u}$ denotes an irrational coding of rotation whose parameters satisfy $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$, then it is proved in [28] that $\mathbf{u}$ is also quasi-Sturmian. The transcendence of the associated continued fraction is thus already shown in Theorem 6, which ends the proof. □

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