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On the transcendence of real numbers with a regular expansion

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Abstract

We apply the Ferenczi–Mauduit combinatorial condition obtained via a reformulation of Ridout's theorem to prove that a real number whose b-ary expansion is the coding of an irrational rotation on the circle with respect to a partition in two intervals is transcendental. We also prove the transcendence of real numbers whose b-ary expansion arises from a non-periodic three-interval exchange transformation.

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1. Introduction

It is well known that, for any integer $b \geqslant 2$, the b-ary expansion of a rational number should be ultimately periodic, but a long-standing problem, apparently asked for the first time by Borel [6], is the following: how regular or random (depending on the viewpoint) is the b-ary expansion of an algebraic irrational number?

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A general conjecture claims that it should be totally random, requiring an algebraic irrational number to be normal in any base $b \geqslant 2$ (i.e., each block of length l occurs with frequency $1/b^l$). Though this conjecture is considered as out of reach, some results have already been known in this direction for more than 70 years [2,5, 8–10,13–15]. These results, proved with different methods, express the following idea: if the b-ary expansion of an irrational number could be obtained by a "too regular" process, then this number is transcendental. This regularity is used in [2,5,8,9] to provide (too) good approximations by rational numbers and in [10,13–15] to obtain functional equations satisfied by certain power series whose coefficients are related to the considered b-ary expansions.

More recently, Ferenczi and Mauduit [11] gave a purely combinatorial condition for transcendence obtained via a reformulation of Ridout's theorem [17]. In particular, they deduce from it the transcendence of the irrational numbers whose expansion has the lowest possible complexity (that is, whose expansion is a Sturmian sequence). This condition is also used in [4] to prove that the binary expansion of an algebraic irrational number cannot be a fixed point of a non-trivial constant-length or primitive morphism and in [18] to obtain the transcendence of real numbers whose b-ary expansion is an Arnoux–Rauzy sequence defined over an alphabet with more than three letters. We refer the reader to the recent survey [3] for more explanations on this subject and on transcendence results related to the continued fraction expansion.

In this paper, we apply the Ferenczi–Mauduit condition to prove the transcendence of real numbers whose b-ary expansion is the coding of an irrational rotation on the circle with respect to a partition in two intervals. This result generalizes in particular those obtained for real numbers with a Sturmian or a quasi-Sturmian expansion in [3,11]. We also prove the transcendence of real numbers whose b-ary expansion arises from a non-periodic three-interval exchange transformation.

2. Definitions and results

2.1. Sequences and morphisms

A finite word w on a finite alphabet $\mathscr{A}$ is an element of the free monoid generated by $\mathscr{A}$ for the concatenation. We denote by $|w|$ the length of the word w, that is, the number of its letters. We call complexity function of a finitely valued sequence $\mathbf{u}$ the function which associates with each integer n the number $p(n)$ of different words of length n occurring in $\mathbf{u}$. An occurrence for a word $w_0 w_1 \dots w_r$ in a sequence $\mathbf{u} = (u_k)_{k \geqslant 0}$ is an integer i such that $u_{i+k} = w_k$ for $0 \leqslant k \leqslant r$. A sequence in which all the factors have an infinite number of occurrences is called recurrent. When for any factor the difference between two consecutive occurrences is bounded, the sequence is called uniformly recurrent. For such a sequence, we say that the word w appears with gaps bounded by k if the difference between two consecutive occurrences of w is at most k.

Let $\mathbf{u}$ be a uniformly recurrent sequence over the alphabet $\mathscr{A}$ and let w be a non-empty factor of $\mathbf{u}$. A return word to w of $\mathbf{u}$ is a factor $u_{[i,k-1]}$ $(= u_i \dots u_{k-1})$ of $\mathbf{u}$ such that i and k are two consecutive occurrences of w. We denote by $\mathscr{R}_{\mathbf{u},w}$ the set of return words to w in $\mathbf{u}$.

A sequence is called Sturmian if $p(n) = n + 1$ for every integer n. More generally, a non-periodic sequence is called quasi-Sturmian (see for instance [7]) if there exists a positive integer c such that $p(n) \leqslant n + c$ for every integer n.

In the following, morphism will mean homomorphism of (free) monoid and $\{x\}$ will mean the fractional part of the real x. A morphism ϕ such that no letter is mapped to the empty word is said to be non-erasing.

2.2. Coding of rotations

Let $(\alpha, \beta) \in (0,1)^2$. The coding of rotation corresponding to the parameters (α, β, x) is the symbolic sequence $\mathbf{u} = (u_n)_{n \geqslant 0}$ defined over the binary alphabet $\{0, 1\}$ by

$$u_n = \begin{cases} 1 & \text{if } \{x + n\alpha\} \in [0, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

When α is rational the sequences obtained are clearly periodic, otherwise the coding of rotation is said irrational. The cases $\beta = \alpha$ or $\beta = 1 - \alpha$ give Sturmian sequences and, more generally, the case $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$ gives quasi-Sturmian sequences (see [19]). A coding of rotation is called non-degenerate if its parameters satisfy: α is irrational and $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$.

2.3. Interval exchange transformations

Let $s \in \mathbb{N}$, $s \geqslant 2$. Let σ be a permutation of the set $\{1, 2, \dots, s\}$ and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ be a vector in $\mathbb{R}^s$ with positive entries. Let

$$I = [0, |\lambda|\,), \quad \text{where } |\lambda| = \sum_{i=1}^{s} \lambda_i \text{ and for } 1 \leqslant i \leqslant s, \quad I_i = \left[\sum_{j<i} \lambda_j, \sum_{j \leqslant i} \lambda_j\right).$$

The interval exchange transformation associated with (λ, σ) is the map E from I into itself, defined as the piecewise isometry which arises from ordering the intervals I_i with respect to σ. More precisely, if $x \in I_i$,

$$E(x) = x + a_i, \quad \text{where } a_i = \sum_{k < \sigma^{-1}(i)} \lambda_{\sigma(k)} - \sum_{k<i} \lambda_k.$$

We can introduce a natural coding of the orbit of a point under the action of an interval exchange transformation by assigning to each element of this orbit the index of the interval which contains it. We say that an interval exchange transformation satisfies the i.d.o.c. (infinite distinct orbit condition) introduced in [12] if the orbits of its discontinuities are infinite and distinct.

Main result. We shall prove the following, extending the result obtained for Sturmian and quasi-Sturmian sequences in [3,11].

Theorem 1. *Let x be a real number and $\mathbf{u} = (u_n)_{n \geqslant 0}$ be its b-ary expansion, where $b \geqslant 2$ is a fixed base. Then the number x is transcendental if one of the following conditions holds*:

- *the sequence* $\mathbf{u}$ *is an irrational coding of rotation*,
- *the sequence* $\mathbf{u}$ *is the natural coding of a non-periodic three-interval exchange transformation.*

Remark 1. If $\mathbf{u}$ is an irrational coding of rotation or the natural coding of a non-periodic interval exchange transformation, then x is an irrational number, $\mathbf{u}$ being not an eventually periodic sequence. Moreover, if $\mathbf{u}$ is a rational coding of rotation or the natural coding of a periodic interval exchange transformation, then x is obviously a rational number.

Theorem 1 is obtained via the following combinatorial translation of a result due to Ridout [17]. We recall that the result of Ridout is an improvement of Roth's theorem [20] (see also [16]).

Theorem 2 (Ferenczi–Mauduit [11]). *Let Θ be an irrational number, such that its b-ary expansion begins, for every integer $n \in \mathbb{N}$, in $0.u_n v_n v_n v'_n$, where u_n is a possibly empty word and where v_n is a non-empty word admitting v'_n as a prefix. If $|v_n|$ tends to infinity,* $\limsup(|u_n|/|v_n|) < \infty$, *and* $\liminf(|v'_n|/|v_n|) > 0$, *then Θ is a transcendental number.*

3. Proof of Theorem 1

In the following, we will say that a sequence $\mathbf{u}$ satisfies property $\mathscr{P}$, if any $\mathbf{v} \in \overline{\mathcal{O}(\mathbf{u})}$ begins, for every integer $n \in \mathbb{N}$, in $u_n v_n v_n v'_n$, where u_n is a possibly empty word, v_n is a non-empty word admitting v'_n as a prefix, $|v_n|$ tends to infinity, $\limsup(|u_n|/|v_n|) < \infty$, and $\liminf(|v'_n|/|v_n|) > 0$. It thus follows from Theorem 2 that if $\mathbf{u}$ satisfies property $\mathscr{P}$ and if x is an irrational number whose b-ary expansion is in $\overline{\mathcal{O}(\mathbf{u})}$, then x is transcendental.

For each integer k, let us introduce the following two morphisms:

$$\begin{array}{ccccccc} & \mathscr{F}_k & & & & \mathscr{G}_k & \\ 1 & \mapsto & 13 & & 1 & \mapsto & 12^k \\ 2 & \mapsto & 2^{k+1}3 & & 2 & \mapsto & 12^{k+1} \\ 3 & \mapsto & 2^k 3 & & 3 & \mapsto & 13. \end{array}$$

Having fixed the above notation, we can give the following combinatorial structure for natural codings of i.d.o.c. three-interval exchanges. This result will play a key role in the proof of Theorem 1.

Theorem 3 (Adamczewski [1]). *Let* $\mathbf{u}$ *be the natural coding of the orbit of* 0 *under the action of an i.d.o.c. three-interval exchange. Then, there exist a non-erasing morphism* ϕ *defined on* $\{1,2,3\}$ *and a sequence* $(a_n,i_n)_{n\geqslant 0}\in(\mathbb{N}\times\{0,1\})^{\mathbb{N}}$, $(a_n)_{n\geqslant 0}$ *not eventually vanishing and* $(i_n)_{n\geqslant 0}$ *not eventually constant, such that*

$$\mathbf{u} = \lim_{n\to\infty} \phi\left(\prod_{j=0}^{n} (\mathscr{F}_{a_j}^{i_j}\circ\mathscr{G}_{a_j}^{1-i_j})(1)\right),$$

where $\prod$ *means the composition of morphisms from left to right.*

Let $\mathbf{u}$ be the natural coding of an i.d.o.c. three-interval exchange. For every non-negative integer k, we introduce (following the notation of Theorem 3) the sequence $\mathbf{v}_k$ defined by

$$\mathbf{v}_k = \lim_{n\to\infty} \prod_{j=k}^{n} (\mathscr{F}_{a_j}^{i_j}\circ\mathscr{G}_{a_j}^{1-i_j})(1).$$

We will write $\mathbf{v}$ instead of $\mathbf{v}_0$ and we will denote by ϕ_k the morphism

$$\phi_k = \prod_{j=0}^{k-1} (\mathscr{F}_{a_j}^{i_j}\circ\mathscr{G}_{a_j}^{1-i_j}).$$

It thus follows that $\mathbf{v} = \phi_k(\mathbf{v}_k)$.

We first state the following result which gives the key idea to show that a sequence obtained by a composition of morphisms satisfies property $\mathscr{P}$.

Lemma 1. *If there exist a pair* (w,w') *of finite words on* $\{1,2,3\}$ *and an increasing sequence of integers* $(k_l)_{l\geqslant 0}$, *such that*:

- *for all* $l\in\mathbb{N}$, *the word* www' *appears in* $\mathbf{v}_{k_l}$ *with bounded gaps, the bound being independent of the integer* l,
- w' *is a prefix of* w *and either the letters* 1 *and* 3 *or the letter* 2 *appear in* w',

then, the sequence $\mathbf{u}$ *satisfies property* $\mathscr{P}$.

Proof. Let (w,w') be a pair of words and let $(k_l)_{l\geqslant 0}$ be an increasing sequence of integers, with the required properties. Let k be an integer and let us note $w_k=\phi_k(w)$ and $w'_k=\phi_k(w')$.

It follows from the definition of $\mathscr{F}_k$ and $\mathscr{G}_k$ and from the condition on the sequence $(a_n,i_n)_{n\geqslant 0}$ that $|w_k|$ tends to infinity with k. Moreover, we easily obtain by induction that for every integer k,

$$|\phi_k(13)|>|\phi_k(2)| = \max\{|\phi_k(1)|,\ |\phi_k(2)|,\ |\phi_k(3)|\}, \tag{1}$$

which implies

$$\frac{|w'_k|}{|w_k|} \geqslant \frac{\min\{|\phi_k(2)|,\ |\phi_k(13)|\}}{|w|\,|\phi_k(2)|} = \frac{1}{|w|} \quad \text{and} \quad \text{thus } \liminf_{k\to\infty} \frac{|w'_k|}{|w_k|} \geqslant \frac{1}{|w|} > 0.$$

Let $\mathbf{w} \in \overline{\mathcal{O}(\mathbf{v})}$. Since $\mathbf{v}$ is uniformly recurrent, for every $l \geqslant 0$ the word $w_{k_l} w_{k_l} w'_{k_l}$ occurs in $\mathbf{w}$ and its first occurrence is at most $R_l = \max\{|u|,\ u \in \mathcal{R}_{\mathbf{v}, w_{k_l} w_{k_l} w'_{k_l}}\}$. Then, it remains to prove that $\limsup_{l\to\infty} \frac{R_l}{|w_{k_l}|} < +\infty$. Moreover, we have that $R_l \leqslant \max\{|\phi_{k_l}(v)|,\ v \in \mathcal{R}_{\mathbf{v}_{k_l}, www'}\}$ and www' appears with bounded gaps in $\mathbf{v}_{k_l}$. The bound being independent of l, there thus exists a positive c (independent of l) such that $\max\{|v|,\ v \in \mathcal{R}_{\mathbf{v}_{k_l}, www'}\} \leqslant c$. This implies that $R_l \leqslant c|\phi_{k_l}(2)|$ and since $|w_{k_l}| \geqslant |\phi_{k_l}(2)|$, it follows that $\limsup_{l\to\infty} \frac{R_l}{|w_{k_l}|} \leqslant c$, concluding the proof. □

We will also need the following modification of Lemma 1, where the condition on w' is relaxed while the one on the sequence $(a_n, i_n)_{n\geqslant 0}$ is strengthened. If $a \in \mathbb{N}$ and $i \in \{0, 1\}$, we will denote by $(a, i)^m$ a block of m consecutive (a, i) occurring in the sequence $(a_n, i_n)_{n\geqslant 0}$.

Lemma 2. *We assume that neither* $(0,0)^3$ *nor* $(0,1)^3$ *appear in* $(a_n, i_n)_{n\geqslant 0}$ *and that* $(a_n)_{n\geqslant 0}$ *is bounded by* 2. *If there exist a pair* (w, w') *of finite words on* $\{1,2,3\}$ *and an increasing sequence of integers* $(k_l)_{l\geqslant 0}$, *such that*:

- *for all* $l \in \mathbb{N}$, *the word* www' *appears in* $\mathbf{v}_{k_l}$ *with bounded gaps, the bound being independent of the integer* l,
- w' *is a non-empty prefix of* w,

then, the sequence $\mathbf{u}$ *satisfies property* $\mathcal{P}$.

Proof. If 1 and 3 or 2 appear in w', then Lemma 1 is enough to conclude. Otherwise, at least 1 or 3 appears in w' since it is a non-empty word.

Let us assume that 1 appears in w'. By hypothesis, there exists c such that the word www' appears in each sequence $\mathbf{v}_{k_l}$ with gaps bounded by c. Moreover, for every integer $l \geqslant 3$, there exists a morphism σ_l, given by a composition of three morphisms of type $\mathcal{F}_k$ and $\mathcal{G}_k$, such that $\mathbf{v}_{k_l - 3} = \sigma_l(\mathbf{v}_{k_l})$. Since $(a_n)_{n\geqslant 0}$ is bounded, the set $\{\sigma_l, l \geqslant 3\}$ is finite and there exists a morphism σ, given by a composition of three morphisms of type $\mathcal{F}_k$ and $\mathcal{G}_k$, such that $\mathbf{v}_{k_l - 3} = \sigma(\mathbf{v}_{k_l})$ for an infinite number of integers l. It thus follows that there exists c' such that the word $\sigma(w)\sigma(w)\sigma(w')$ appears with gaps bounded by c' in an infinite number of sequences $\mathbf{v}_k$. But, since $(0,0)^3$ is not allowed, σ is not equal to $\mathcal{G}_0^3$, implying that 13 or 12 appears in $\sigma(1)$ and therefore in $\sigma(w')$. The pair $(\sigma(w), \sigma(w'))$ thus satisfies the condition required in Lemma 1, hence the result.

If we assume that 3 appears in w', we can do the same reasoning, applying, instead of $\mathcal{G}_0^3$, that $\mathcal{F}_0^3$ is not allowed. □

The next step consists in studying the combinatorial structure of sequences obtained by a composition of morphisms as in Theorem 3. More precisely, we have to show that such sequences satisfy property $\mathscr{P}$ and this result will be obtained via Lemmas 1 and 2. Next, we will easily deduce Theorem 1 from Lemma 3.

Lemma 3. *Let* $(a_n, i_n)_{n\geqslant 0} \in (\mathbb{N} \times \{0,1\})^{\mathbb{N}}$, *with* $(a_n)_{n\geqslant 0}$ *not eventually vanishing and with* $(i_n)_{n\geqslant 0}$ *not eventually constant, and let* $\mathbf{v}$ *be the infinite sequence defined by*

$$\mathbf{v} = \lim_{n\to\infty} \left(\prod_{j=0}^{n} (\mathscr{F}_{a_j}^{i_j} \circ \mathscr{G}_{a_j}^{1-i_j})(1) \right).$$

Then, the sequence $\mathbf{v}$ *satisfies property* $\mathscr{P}$.

Proof. We keep in this proof the notations introduced in the beginning of this section.

Let us first assume that the block $(0,0)^3$ appears infinitely often in the sequence $(a_n, i_n)_{n\geqslant 0}$. Then, at least one of the following holds:

(a) there exists $j \in \mathbb{N}^*$ such that $(j,0)(0,0)^3$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$,
(b) there exists $j \in \mathbb{N}$ such that $(j,1)(0,0)^3$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$,
(c) there exists an increasing sequence of integers $(j_m)_{m\geqslant 0}$, $j_m \geqslant 3$, such that for every m, the block $(j_m, 0)(0,0)^3$ appears in $(a_n, i_n)_{n\geqslant 0}$,
(d) there exists an increasing sequence of integers $(j_m)_{m\geqslant 0}$, $j_m \geqslant 3$, such that for every m, the block $(j_m, 1)(0,0)^3$ appears in $(a_n, i_n)_{n\geqslant 0}$.

(a) In this case, we obtain that for an infinite number of k, $\mathbf{v}_k = (\mathscr{G}_j \circ \mathscr{G}_0^3)(\mathbf{v}_{k+4})$, which implies that $(12^j)^3$ appears in $\mathbf{v}_k$ with gaps bounded by $3j+5$. Indeed, since

$$\begin{array}{ccc} & \mathscr{G}_j \circ \mathscr{G}_0^3 & \\ 1 & \mapsto & 12^j \\ 2 & \mapsto & (12^j)^3 12^{j+1} \\ 3 & \mapsto & (12^j)^3 13, \end{array} \qquad (2)$$

the return words to $(12^j)^3$ in $\mathbf{v}_k$ are exactly 12^j, $(12^j)^3 2$ and $(12^j)^3 13$. Therefore, the sequence $\mathbf{v}$ satisfies $\mathscr{P}$ in view of Lemma 1 since $j \geqslant 1$.

(b) We obtain as above that for an infinite number of k, $\mathbf{v}_k = (\mathscr{F}_j \circ \mathscr{G}_0^3)(\mathbf{v}_{k+4})$. This implies that $(13)^3$ appears in $\mathbf{v}_k$ with gaps bounded by $j+8$. Indeed, since

$$\begin{array}{ccc} & \mathscr{F}_j \circ \mathscr{G}_0^3 & \\ 1 & \mapsto & 13 \\ 2 & \mapsto & (13)^3 2^{j+1} 3 \\ 3 & \mapsto & (13)^3 2^j 3, \end{array} \tag{3}$$

the return words to $(13)^3$ are 13, $(13)^3 2^j 3$ and $(13)^3 2^{j+1} 3$. Therefore, the sequence $\mathbf{v}$ satisfies $\mathscr{P}$ in view of Lemma 1.

(c) In this case, we obtain that for every integer m there exists an integer k such that $\mathbf{v}_k = (\mathscr{G}_{j_m} \circ \mathscr{G}_0^3)(\mathbf{v}_{k+4})$ and thus, in view of (2) and since $j_m \geqslant 3$, the word 2^3 appears with gaps bounded by 6 in $\mathbf{v}_k$ for an infinite number of integers k, hence $\mathbf{v}$ satisfies $\mathscr{P}$.

(d) In this case, we obtain that for every integer m there exists an integer k such that $\mathbf{v}_k = (\mathscr{F}_{j_m} \circ \mathscr{G}_0^3)(\mathbf{v}_{k+4})$ and thus, in view of (3) and since $j_m \geqslant 3$, for an infinite number of integers k, every factor of $\mathbf{v}_k$ of length greater than 8 contains either $(13)^3$ or 2^3. It follows that every factor of $\mathbf{v}$ of length greater than $10|\phi_k(2)| - 2$ contains either $(\phi_k(13))^3$ or $(\phi_k(2))^3$, which implies following (1) that $\mathbf{v}$ satisfies $\mathscr{P}$.

The case when the block $(0,1)^3$ occurs infinitely often could be dealt with as above using the symmetry between $\mathscr{F}_k$ and $\mathscr{G}_k$.

Now, we can assume without restriction that neither $(0,0)^3$ nor $(0,1)^3$ appear in $(a_n, i_n)_{n \geqslant 0}$ since the conditions required to satisfy $\mathscr{P}$ are clearly preserved by non-erasing morphism. This directly implies that the words 1^3 and 3^3 cannot appear in any sequence $\mathbf{v}_k$.

Let us assume that there exists an increasing sequence $(k_l)_{l \geqslant 0}$ such that $j_l = a_{k_l} \geqslant 3$. Then, $\mathbf{v}_{k_l} = \mathscr{F}_{j_l}(\mathbf{v}_{k_l+1})$ or $\mathbf{v}_{k_l} = \mathscr{G}_{j_l}(\mathbf{v}_{k_l+1})$. Since $j_l \geqslant 3$ and 1^3 and 3^3 are not factors of $\mathbf{v}_{k_l+1}$, we obtain that 2^3 appears in $\mathbf{v}_{k_l}$ with gaps bounded by 8, hence $\mathbf{v}$ satisfies $\mathscr{P}$.

Now, we can assume without restriction that neither $(0,0)^3$ nor $(0,1)^3$ appear in $(a_n, i_n)_{n \geqslant 0}$ and that $(a_n)_{n \geqslant 0}$ is bounded by 2. This implies in particular that we can use Lemma 2.

Let us assume that $(0,0)^2$ appears infinitely often in $(a_n, i_n)_{n \geqslant 0}$. Then, for an infinite number of integers k, either $\mathbf{v}_k = (\mathscr{G}_j \circ \mathscr{G}_0^2)(\mathbf{v}_{k+3})$ with $1 \leqslant j \leqslant 2$, or $\mathbf{v}_k = (\mathscr{F}_j \circ \mathscr{G}_0^2)(\mathbf{v}_{k+3})$ with $0 \leqslant j \leqslant 2$, since $(a_n)_{n \geqslant 0}$ is bounded by 2.

In the first case, the word $(12^j)^2 1$ appears in $\mathbf{v}_k$ with gaps bounded by $3j + 4$ ($\leqslant 10$ because j is at most 2) and

$$\begin{array}{ccc} & \mathscr{G}_j \circ \mathscr{G}_0^2 & \\ 1 & \mapsto & 12^j \\ 2 & \mapsto & (12^j)^2 12^{j+1} \\ 3 & \mapsto & (12^j)^2 13, \end{array}$$

hence $\mathbf{v}$ satisfies $\mathscr{P}$ in view of Lemma 2.

In the second case, the word 31313 appears in $\mathbf{v}_k$ with gaps bounded by $j+6$ ($\leqslant 8$ because j is at most 2). We obtain that $\mathbf{v}$ satisfies $\mathscr{P}$ in view of Lemma 2 (here of course $w = 31$ and $w' = 3$).

The case where $(0,1)^2$ appears infinitely often is similar.

We can thus assume without restriction that neither $(0,0)^2$ nor $(0,1)^2$ does appear in $(a_n, i_n)_{n\geqslant 0}$ and that $(a_n)_{n\geqslant 0}$ is bounded by 2.

Let us assume that $((0,1)(0,0))^2$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$. Then, for an infinite number of integers k, $\mathbf{v}_k = (\mathscr{F}_0 \circ \mathscr{G}_0)^2(\mathbf{v}_{k+4})$. Since

$$\begin{array}{ccc} & (\mathscr{F}_0 \circ \mathscr{G}_0)^2 & \\ 1 & \mapsto & 13133 \\ 2 & \mapsto & 131331323133 \\ 3 & \mapsto & 13133133, \end{array}$$

the word 31313 appears with bounded gaps in an infinite number of sequences $\mathbf{v}_k$. Lemma 2 thus implies that $\mathbf{v}$ satisfies $\mathscr{P}$.

The case where $((0,0)(0,1))^2$ appears infinitely often could be dealt with as above.

We can thus assume without restriction that $(a_n)_{n\geqslant 0}$ is bounded by 2 and that $(a_n)_{n\geqslant 0}$ does not take consecutively more than three times the value 0. This together with the fact that 1^3 and 3^3 do not appear in any $\mathbf{v}_k$ implies the existence of c such that the letter 2 appears with gaps bounded by c in $\mathbf{v}_k$ for every integer k. Indeed, if $\mathbf{w}$ is a sequence in which 1^3 and 3^3 do not appear and if j denotes a positive integer, then 2 appears with gaps bounded by 6 in $\mathscr{F}_j(\mathbf{w})$ and $\mathscr{G}_j(\mathbf{w})$.

Let us assume that $(2,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$. Then, for an infinite number of integers k, $\mathbf{v}_k = \mathscr{G}_2(\mathbf{v}_{k+1})$. Since 2^3 appears in $\mathscr{G}_2(2)$, it thus follows that $\mathbf{v}$ satisfies $\mathscr{P}$.

The case where $(2,1)$ appears infinitely often could be dealt with as above.

We can thus assume without restriction that $(a_n)_{n\geqslant 0}$ is bounded by 1 and the existence of c such that the letter 2 appears with gaps bounded by c in $\mathbf{v}_k$ for every integer k.

Let us assume that $(1,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$. Then, at least one of the following holds:

(a) the block $(1,0)(1,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$ and then, for an infinite number of integers k, $\mathbf{v}_k = (\mathscr{G}_1)^2(\mathbf{v}_{k+2})$, implying that the word 2122122 (which is a factor of $\mathscr{G}_1^2(2)$) occurs with uniformly bounded gaps in infinitely many $\mathbf{v}_k$,

(b) the block $(1,1)(1,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$ and then, for an infinite number of integers k, $\mathbf{v}_k = (\mathscr{F}_1 \circ \mathscr{G}_1)(\mathbf{v}_{k+2})$, implying that the word 3223223 (which is a factor of $(\mathscr{F}_1 \circ \mathscr{G}_1)(2)$) occurs with uniformly bounded gaps in infinitely many $\mathbf{v}_k$,

(c) the block $(0,1)(1,0)$ appears infinitely often in $(a_n, i_n)_{n\geqslant 0}$ and then, for an infinite number of integers k, $\mathbf{v}_k = (\mathscr{F}_0 \circ \mathscr{G}_1)(\mathbf{v}_{k+2})$, implying that the word 32323 (which is a factor of $(\mathscr{F}_0 \circ \mathscr{G}_1)(2)$) occurs with uniformly bounded gaps in infinitely many $\mathbf{v}_k$,

(d) the block $(0,0)(1,0)$ appears infinitely often in $(a_n, i_n)_{n \geqslant 0}$ and then, for an infinite number of integers k, $\mathbf{v}_k = (\mathscr{G}_0 \circ \mathscr{G}_1)(\mathbf{v}_{k+2})$, implying that the word 12121 occurs with uniformly bounded gaps in infinitely many $\mathbf{v}_k$. Indeed, the word 1212 is a factor of $(\mathscr{G}_0 \circ \mathscr{G}_1)(2)$ and it is always followed by the letter 1.

In each case, Lemma 2 implies that $\mathbf{v}$ satisfies property $\mathscr{P}$.

The case where $(1,1)$ appears infinitely often in $(a_n, i_n)_{n \geqslant 0}$ is similar and this finishes the proof of Lemma 3. □

Proof of Theorem 1. We first should recall (see Remark 1) that the symbolic sequences we consider are not eventually periodic. This implies that all the real numbers concerned with Theorem 1 are irrational.

If $\mathbf{u}$ denotes the natural coding of the orbit of 0 under the action of an i.d.o.c. three-interval exchange, then Theorem 3 together with Lemma 3 implies the existence of a non-erasing morphism ϕ defined on $\{1,2,3\}$ such that $\mathbf{u} = \phi(\mathbf{v})$, the sequence $\mathbf{v}$ satisfying property $\mathscr{P}$. It follows immediately that $\mathbf{u}$ satisfies $\mathscr{P}$ too and then, the natural coding of the orbit of any point satisfies the condition required in Theorem 2, concluding the proof in this case. If $\mathbf{u}$ denotes the natural coding of a non-periodic three-interval exchange which does not satisfy the i.d.o.c., then it is shown in [1] that $\mathbf{u}$ must be quasi-Sturmian and thus the result is already proved in [3].

If $\mathbf{u}$ denotes a non-degenerate coding of rotation of parameters $(\alpha, \beta, 0)$, then it is shown in [1] that there exist a natural coding of the orbit of 0 under the action of an i.d.o.c. three-interval exchange $\mathbf{v}$ and a non-erasing morphism ϕ from $\{1,2,3\}$ into $\{1,2\}$ such that either $\mathbf{u} = \phi(\mathbf{v})$ or $\mathbf{u} = 1S(\phi(\mathbf{v}))$, where S denotes the classical shift transformation. In these two cases, we easily obtain that the sequence $\mathbf{u}$ satisfies $\mathscr{P}$, since it is the case for $\mathbf{v}$ and then, any coding of rotation of parameters (α, β, x) satisfies the conditions of Theorem 2, concluding the proof in this case. Finally, if $\mathbf{u}$ denotes an irrational coding of rotation whose parameters satisfy $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$, then it is proved in [19] that $\mathbf{u}$ is also quasi-Sturmian, which ends the proof. □

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