ADDENDUM TO: MAHLER'S METHOD IN SEVERAL VARIABLES AND FINITE AUTOMATA

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The aim of this note is to prove the following extension of one of the main results of [2] concerning the algebraic independence of values of M-functions at multiplicatively independent algebraic points. We retain the notations introduced in [2].

Theorem A.1. *Let $r \geq 1$ be an integer and $\mathbb{K} \subseteq \overline{\mathbb{Q}}$ be a field. For every integer i, $1 \leq i \leq r$, we let $q_i \geq 2$ be an integer, $f_i(z) \in \mathbb{K}[[z]]$ be an M_{q_i}-function, and $\alpha_i \in \mathbb{K}$, $0 < |\alpha_i| < 1$, be such that $f_i(z)$ is well-defined at α_i. Let us assume that the numbers $\alpha_1, \ldots, \alpha_r$ are pairwise multiplicatively independent. Then $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them belongs to $\mathbb{K}$.*

Theorem A.1 strengthens part (i) of [2, Theorem 1.1] in which a stronger condition was required: the points α_i had to be (globally) multiplicatively independent and not just pairwise multiplicatively independent. For instance, assuming that $f_1(z), f_2(z)$ and $f_3(z)$ are M-functions that take transcendental values at $\frac{1}{2}, \frac{1}{5}$ and $\frac{1}{10}$ respectively, Theorem A.1 implies that these three numbers are algebraically independent, while [2, Theorem 1.1] could not apply.

We deduce from Theorem A.1 the following generalization of [2, Theorem 2.3].

Theorem A.2. *Let $r \geq 1$ be an integer. Let $b_1, \ldots, b_r$ be pairwise multiplicatively independent positive integers, and, for every i, $1 \leq i \leq r$, let x_i be a real number that is automatic in base b_i. Then the numbers $x_1, \ldots, x_r$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them is rational.*

We omit the proof of Theorem A.2 as it can be deduced from Theorem A.1, just as [2, Theorem 2.3] can be deduced from [2, Theorem 1.1].

The rest of this note is devoted to the proof of Theorem A.1. As with the proof of [2, Theorem 1.1], it mainly relies on some of the general results concerning Mahler's method in several variables proved in [2] (e.g., Corollary 3.5, Corollary 3.9, and Theorem 5.9). The main novelty is the use of a trick introduced by Loxton and van der Poorten [3] in this framework to deal with values of Mahler functions at certain points with multiplicatively dependent coordinates.

1. Proof of Theorem A.1

In order to prove Theorem A.1, we first need three auxiliary results.

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1.1. **Auxiliary results.** Our first auxiliary result is a lemma concerning algebraic numbers, on which Loxton and van der Poorten's trick is based.

Lemma A.3. *Let $\alpha_1, \ldots, \alpha_r \in \overline{\mathbb{Q}}$ be algebraic numbers such that $0 < |\alpha_i| < 1$ for every i, $1 \leq i \leq r$. Then there exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_t \in \overline{\mathbb{Q}}$, $0 < |\beta_j| < 1$, $1 \leq j \leq t$, roots of unity $\zeta_1, \ldots, \zeta_r$, and nonnegative integers $\mu_{i,j}$, $1 \leq i \leq r$, $1 \leq j \leq t$, such that*

$$\alpha_i = \zeta_i \prod_{j=1}^{t} \beta_j^{\mu_{i,j}}, \quad \forall i,\, 1 \leq i \leq r\,.$$

Proof. This is [3, Lemma 3] (see also [4, Lemma 3.4.9]). ☐

Our second auxiliary result is the following result about M-functions.

Lemma A.4. *Let $q \geq 2$ be an integer, $f(z)$ be an M_q-function and ζ be a root of unity. Then $f(\zeta z)$ is also an M_q-function.*

Proof. We first recall that the set of M_q-functions is a ring containing $\overline{\mathbb{Q}}(z) \cap \overline{\mathbb{Q}}[[z]]$ and that, given any positive integer ℓ, a power series is an M_q-function if and only if it is an M_{q^ℓ}-function. Let k be such that $\zeta_0 := \zeta^{q^k}$ has order coprime with q. Then there exists a positive integer ℓ such that $\zeta_0^{q^\ell} = \zeta_0$. Since $f(z)$ is also an M_{q^ℓ}-function, we deduce that $f(\zeta_0 z)$ is an M_{q^ℓ}-function and hence an M_q-function. The same argument applies to any power of ζ_0, so that $f(\zeta_0^i z)$ is an M_q-function for every integer $i \geq 0$. Given a positive integer j, substituting z with $z^{q^{jk}}$ and taking $i := q^{k(j-1)}$, we thus deduce that $f((\zeta z)^{q^{jk}})$ is an M_q-function. Now, substituting ζz to z in the minimal q^k-Mahler equation satisfied by $f(z)$, we can write $f(\zeta z)$ as a linear combination over $\overline{\mathbb{Q}}(z)$ of the series $f((\zeta z)^{q^{jk}})$, $j \in \{1, \ldots, r\}$, where r is the order of this minimal equation. Since $f(\zeta z)$ is a power series, we can ensure that $f(\zeta z)$ can in fact be written as a linear combination over $\overline{\mathbb{Q}}(z) \cap \overline{\mathbb{Q}}[[z]]$ of some M_q-functions. It therefore follows that $f(\zeta z)$ is an M_q-function, as wanted. ☐

Our third auxiliary result is about algebraic independence of power series.

Lemma A.5. *Let r and t be two positive integers, $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r \in \mathbb{N}^t$ be vectors that are pairwise linearly independent over $\mathbb{Q}$, and, for every i, $1 \leq i \leq r$, let m_i be a positive integer and $f_{i,1}(z), \ldots, f_{i,m_i}(z) \in \overline{\mathbb{Q}}[[z]]$. Let $\boldsymbol{z} := (z_1, \ldots, z_t)$ be a vector of indeterminates. Then*

$$\begin{aligned}\operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}&(f_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i}) \,:\, 1 \leq i \leq r,\, 1 \leq j \leq m_i) \\ &= \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_{i,j}(z) \,:\, 1 \leq j \leq m_i)\,.\end{aligned}$$

We recall that $\boldsymbol{z}^{\boldsymbol{\mu}_j} := \prod_{i=1}^{t} z_i^{\mu_{i,j}}$. In order to prove Lemma A.5, we first need to establish a simple result about cones in $\mathbb{R}^t$. We define the convex cone $\mathcal{C}$ spanned by some vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r \in \mathbb{R}^t$ as the set

$$\mathcal{C} := \{a_1\boldsymbol{\mu}_1 + \cdots + a_r\boldsymbol{\mu}_r \;:\; a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}\}\,.$$

A basis of $\mathcal{C}$ is a minimal set of vectors in $\mathbb{R}^t$ such that the convex cone spanned by these vectors is $\mathcal{C}$.

Lemma A.6. *Let $\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r \in \mathbb{N}^t$ be pairwise linearly independent over $\mathbb{Q}$ and $\mathcal{C}$ denote the convex cone spanned by $\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r$. Let us assume that $\{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_s\}$ is a basis of $\mathcal{C}$, for some $1 \leq s \leq r$. Then, $\boldsymbol{\mu}_1$ does not belong to the convex cone $\mathcal{C}^\circ$ spanned by $\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_r$. Furthermore, for any $\boldsymbol{\lambda} \in \mathbb{N}^t$ and any finite set $\Gamma \subset \mathbb{N}^t$, the intersection*

$$(\boldsymbol{\lambda} + \mathbb{N}\boldsymbol{\mu}_1)\bigcap(\Gamma + \mathcal{C}^\circ)$$

is finite.

Proof. Let us start with the first part of the proof. We first note that, since the vector $\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r$ are pairwise linearly independent over $\mathbb{Q}$, $\boldsymbol{\mu}_1$ is a nonzero vector. By assumption, for every i, $s < i \leq r$, there exist nonnegative real numbers $\lambda_{i,j}$, $1 \leq j \leq s$, such that

$$\boldsymbol{\mu}_i = \sum_{j=1}^{s} \lambda_{i,j}\boldsymbol{\mu}_j\,. \tag{1.1}$$

Let us assume by contradiction that $\boldsymbol{\mu}_1$ belongs to the convex cone spanned by $\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_r$. Then, there exist nonnegative real numbers $\theta_2,\ldots,\theta_r$ such that

$$\boldsymbol{\mu}_1 = \sum_{j=2}^{r} \theta_j\boldsymbol{\mu}_j\,. \tag{1.2}$$

We deduce from (1.1) and (1.2) that

$$\begin{aligned}\boldsymbol{\mu}_1 &= \sum_{j=2}^{s}\theta_j\boldsymbol{\mu}_j + \sum_{i=s+1}^{r}\theta_i\sum_{j=1}^{s}\lambda_{i,j}\boldsymbol{\mu}_j \\ &= \left(\sum_{i=s+1}^{r}\theta_i\lambda_{i,1}\right)\boldsymbol{\mu}_1 + \sum_{j=2}^{s}\left(\theta_j + \sum_{i=s+1}^{r}\theta_i\lambda_{i,j}\right)\boldsymbol{\mu}_j\end{aligned}$$

and hence

$$\left(1 - \sum_{i=s+1}^{r}\theta_i\lambda_{i,1}\right)\boldsymbol{\mu}_1 = \sum_{i=2}^{s}\left(\theta_j + \sum_{i=s+1}^{r}\theta_i\lambda_{i,j}\right)\boldsymbol{\mu}_j\,.$$

On the one hand, if $1 - \sum_{i=s+1}^{r}\theta_i\lambda_{i,1} > 0$, then $\boldsymbol{\mu}_1$ would belong to the convex cone generated by $\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_s$, which would contradict the fact that $\{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_s\}$ is a basis of $\mathcal{C}$. On the other hand, if $1 - \sum_{i=s+1}^{r}\theta_i\lambda_{i,1} < 0$, since $\boldsymbol{\mu}_1 \neq 0$, at least one of the coordinates of $\boldsymbol{\mu}_1$ would be negative, which is impossible. Hence $1 - \sum_{i=s+1}^{r}\theta_j\lambda_{i,1} = 0$ and we deduce that

$$\theta_j + \sum_{i=s+1}^{r}\theta_i\lambda_{i,j} = 0, \quad \forall j,\ 2 \leq j \leq s\,. \tag{1.3}$$

Since all these numbers are nonnegative, we first observe that $\theta_j = 0$, for every $j \in \{2,\ldots,s\}$. Since $\boldsymbol{\mu}_1$ is nonzero, we infer from (1.2) the existence of $i_0 > s$ such that $\theta_{i_0} \neq 0$. Then, we deduce from (1.3) that $\lambda_{i_0,j} = 0$ for every $j \in \{2,\ldots,s\}$. Thus, it follows from (1.1) that $\boldsymbol{\mu}_{i_0} = \lambda_{i_0,1}\boldsymbol{\mu}_1$, providing a contradiction with the fact that $\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_r$ are pairwise linearly independent over $\mathbb{Q}$. This concludes the first part of the proof.

We now turn to the second part. Let $\boldsymbol{\lambda} \in \mathbb{N}^t$ and Γ be a finite subset of $\mathbb{N}^t$. Let $d := \inf_{\boldsymbol{\kappa} \in \mathcal{C}^\circ} |\boldsymbol{\mu}_1 - \boldsymbol{\kappa}|$ denote the distance between $\boldsymbol{\mu}_1$ and $\mathcal{C}^\circ$. Since we just proved that $\boldsymbol{\mu}_1$ does not belong to $\mathcal{C}^\circ$, we easily deduce that $d > 0$. Set

$$B := \max\{|\boldsymbol{\gamma}| + |\boldsymbol{\lambda}| : \boldsymbol{\gamma} \in \Gamma\}\,.$$

Let $k \in \mathbb{N}$ be such that $\boldsymbol{\lambda} + k\boldsymbol{\mu}_1 \in \Gamma + \mathcal{C}^\circ$. Then

$$\boldsymbol{\lambda} + k\boldsymbol{\mu}_1 = \boldsymbol{\gamma} + \boldsymbol{\mu}\,,$$

for some $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{\mu} \in \mathcal{C}^\circ$. Since $\boldsymbol{\mu}/k \in \mathcal{C}^\circ$, it follows that

$$\frac{B}{k} \geq \frac{|\boldsymbol{\gamma} - \boldsymbol{\lambda}|}{k} = \left|\boldsymbol{\mu}_1 - \frac{\boldsymbol{\mu}}{k}\right| \geq d$$

and hence $k \leq B/d$. We deduce that

$$(\boldsymbol{\lambda} + \mathbb{N}\boldsymbol{\mu}_1) \cap (\Gamma + \mathcal{C}^\circ) \subset \{\boldsymbol{\lambda} + k\boldsymbol{\mu}_1 \,:\, 0 \leq k \leq B/d\}\,.$$

In particular, it is a finite set. □

Proof of Lemma A.5. We argue by induction on r. When $r = 1$, there is nothing to prove. We now assume that $r \geq 2$ and that the result is proven for $r - 1$. Up to reordering the indices, we can assume that $\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_s\}$ is a basis of the cone $\mathcal{C}$ spanned by $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$, for some $s \leq r$. According to our induction hypothesis, we only have to prove that

$$\begin{aligned}\operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}&(f_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i}) \,:\, 1 \leq i \leq r,\, 1 \leq j \leq m_i)\\ &= \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_{1,j}(z) \,:\, 1 \leq j \leq m_1)\\ &\quad + \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i}) \,:\, 2 \leq i \leq r,\, 1 \leq j \leq m_i)\,.\end{aligned}$$

We are going to prove the following stronger fact: for any $g_1(z), \ldots, g_m(z) \in \overline{\mathbb{Q}}[[z]]$ that are linearly independent over $\overline{\mathbb{Q}}(z)$, the power series

$$g_1(\boldsymbol{z}^{\boldsymbol{\mu}_1}), \ldots, g_m(\boldsymbol{z}^{\boldsymbol{\mu}_1}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]]$$

are linearly independent over the ring $\mathbb{A} := \overline{\mathbb{Q}}[[\boldsymbol{z}^{\boldsymbol{\mu}_2}, \ldots, \boldsymbol{z}^{\boldsymbol{\mu}_r}]][\boldsymbol{z}]$.

Let $g_1(z), \ldots, g_m(z) \in \overline{\mathbb{Q}}[[z]]$ be linearly independent over $\overline{\mathbb{Q}}(z)$ and let us assume by contradiction that the series $g_1(\boldsymbol{z}^{\boldsymbol{\mu}_1}), \ldots, g_m(\boldsymbol{z}^{\boldsymbol{\mu}_1})$ are linearly dependent over $\mathbb{A}$. Then, there exist $h_1(\boldsymbol{z}), \ldots, h_m(\boldsymbol{z}) \in \mathbb{A}$, not all zero, such that

$$h_1(\boldsymbol{z})g_1(\boldsymbol{z}^{\boldsymbol{\mu}_1}) + \cdots + h_m(\boldsymbol{z})g_m(\boldsymbol{z}^{\boldsymbol{\mu}_1}) = 0\,. \tag{1.4}$$

Let $\mathcal{C}^\circ$ denote the convex cone spanned by $\boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_r$. By definition of $\mathbb{A}$, there exists a finite set $\Gamma \subset \mathbb{N}^t$ such that the support of each $h_i(\boldsymbol{z})$ is included in $\Gamma + \mathcal{C}^\circ$. Thus, we can write

$$h_i(\boldsymbol{z}) = \sum_{\boldsymbol{\kappa} \in \Gamma + \mathcal{C}^\circ} h_{i,\boldsymbol{\kappa}} \boldsymbol{z}^{\boldsymbol{\kappa}}, \qquad \forall i, 1 \leq i \leq m.$$

We also set $h_{i,\boldsymbol{\kappa}} := 0$ when $\boldsymbol{\kappa} \notin \Gamma + \mathcal{C}^\circ$. Considering the equivalence relation on $\mathbb{N}^t$ defined by $\boldsymbol{\lambda}_1 \sim \boldsymbol{\lambda}_2$ if $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \in \mathbb{Z}\boldsymbol{\mu}_1$, we can defined a set $\Lambda \subset \mathbb{N}^t$ by picking the vector of smallest norm in each equivalence class, so that $\mathbb{N}^t$ can be written as the disjoint union $\bigsqcup_{\boldsymbol{\lambda} \in \Lambda} (\boldsymbol{\lambda} + \mathbb{N}\boldsymbol{\mu}_1)$. For every $\boldsymbol{\lambda} \in \Lambda$, set $\Gamma_{\boldsymbol{\lambda}} := (\Gamma + \mathcal{C}^\circ) \cap (\boldsymbol{\lambda} + \mathbb{N}\boldsymbol{\mu}_1)$. It follows from Lemma A.6 that all the sets $\Gamma_{\boldsymbol{\lambda}}$ are finite. Since the sets $\Gamma_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \in \Lambda$, form a partition of $\Gamma + \mathcal{C}^\circ$, and since every element of $\Gamma_{\boldsymbol{\lambda}}$ can be written $\boldsymbol{\lambda}+n\boldsymbol{\mu}_1$ for some $n\in\mathbb{N}$, we have a decomposition of the form

$$h_i(\boldsymbol{z})=\sum_{\boldsymbol{\lambda}\in\Lambda}\boldsymbol{z}^{\boldsymbol{\lambda}}a_{i,\boldsymbol{\lambda}}(\boldsymbol{z}^{\boldsymbol{\mu}_1}),\quad \forall i, 1\le i\le m\,,$$

where $a_{i,\boldsymbol{\lambda}}(z):=\sum_{n=0}^{\infty}h_{i,\boldsymbol{\lambda}+n\boldsymbol{\mu}_1}z^n$. Since all the sets $\Gamma_{\boldsymbol{\lambda}}$ are finite, the $a_{i,\boldsymbol{\lambda}}(z)$ are in fact polynomials. Since the sets $\boldsymbol{\lambda}+\mathbb{N}\boldsymbol{\mu}_1$, $\boldsymbol{\lambda}\in\Lambda$, are disjoints, identifying the powers of $\boldsymbol{z}$ that belong to $\boldsymbol{\lambda}+\mathbb{N}\boldsymbol{\mu}_1$ in (1.4) leads to

$$\sum_{i=1}^{m}a_{i,\boldsymbol{\lambda}}(z)g_i(z)=0\,,\qquad \forall\boldsymbol{\lambda}\in\Lambda\,.$$

Since the power series $g_1(z),\dots,g_m(z)$ are linearly independent over $\overline{\mathbb{Q}}(z)$, we deduce that $a_{i,\boldsymbol{\lambda}}(z)=0$ for every pair $(i,\boldsymbol{\lambda})\in\{1,\dots,m\}\times\Lambda$. Hence $h_i(\boldsymbol{z})=0$ for all $i\in\{1,\dots,m\}$, which provides a contradiction. □

1.2. **Existence of a suitable linear Mahler system.** The following proposition ensures the existence of suitable linear Mahler systems in several variables that will be used to deduce Theorem A.1 from the main results of [2].

Proposition A.7. *Let $q\ge 2$ be an integer, $\alpha_1,\dots,\alpha_r\in\overline{\mathbb{Q}}$ be pairwise multiplicatively independent, $0<|\alpha_i|<1$ and, for every i, $1\le i\le r$, $f_i(z)\in\overline{\mathbb{Q}}[[z]]$ be an M_q-function that is well defined at α_i. Then there exist a positive integer t, a positive integer ℓ, a point $\boldsymbol{\beta}\in\overline{\mathbb{Q}}^t$, a matrix $T\in\mathcal{M}_t(\mathbb{N})$, some vectors $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r\in\mathbb{N}^t$, and for every i, $1\le i\le r$, roots of unity ζ_i, a positive integer m_i and some M_q-functions $g_{i,1}(z),\dots,g_{i,m_i}(z)\in\overline{\mathbb{Q}}[[z]]$ such that the following hold.*

- (a) *For every $i\in\{1,\dots,r\}$, $\alpha_i=\zeta_i\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$.*
- (b) *For every $i\in\{1,\dots,r\}$, $f_i(\alpha_i)=g_{i,1}(\boldsymbol{\beta}^{\boldsymbol{\mu}_i})$.*
- (c) *For every $i\in\{1,\dots,r\}$, $g_{i,1}(z),\dots,g_{i,m_i}(z)$ are related by a q^ℓ-Mahler system and $\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$ is regular w.r.t. this system.*
- (d) *The functions $g_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i})$, $1\le i\le r$, $1\le j\le m_i$ are related by a T-Mahler system, where $\boldsymbol{z}=(z_1,\dots,z_t)$ is a vector of indeterminates.*
- (e) *The pair $(T,\boldsymbol{\beta})$ is admissible and the point $\boldsymbol{\beta}$ is regular w.r.t. this system.*
- (f) *The spectral radius of T is equal to q^ℓ.*
- (g) *The vectors $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r$ are pairwise linearly independent over $\mathbb{Q}$.*

Proof. We first infer from Lemma A.3 the existence of a positive integer t, multiplicatively independent algebraic numbers $\beta_1,\dots,\beta_t$, $0<|\beta_j|<1$, $1\le j\le t$, roots of unity $\zeta_1,\dots,\zeta_r$ and nonnegative integers $\mu_{i,j}$, $1\le i\le r$, $1\le j\le t$, such that

$$\alpha_i=\zeta_i\prod_{j=1}^{t}\beta_j^{\mu_{i,j}},\qquad \forall i,\ 1\le i\le r\,.$$

Setting $\boldsymbol{\beta}:=(\beta_1,\dots,\beta_t)$ and $\boldsymbol{\mu}_i:=(\mu_{i,1},\dots,\mu_{i,t})$, we get that (a) is satisfied. By Lemma A.4, each $f_i(\zeta_i z)$ is an M_q-function. Applying [2, Lemma 11.1] to the functions $f_i(\zeta_i z)$ and the points $\zeta_i^{-1}\alpha_i=\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$, we can find, for every $i\in\{1,\dots,r\}$, some M_q-functions $g_{i,1}(z),\dots,g_{i,m_i}(z)$ related by some

q^{ℓ_i}-Mahler system with respect to which $\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$ is a regular point and such that $g_{i,1}(\boldsymbol{\beta}^{\boldsymbol{\mu}_i}) = f_i(\alpha_i)$, so that (b) holds. Iterating each one of these systems an appropriate number of times if necessary, we can further assume that the integers ℓ_i, $1 \leq i \leq r$, are all equal to some common integer, say ℓ. Hence (c) is satisfied. Let $A_1(z), \ldots, A_r(z)$ denote the matrices associated with each of these Mahler systems. Let $\boldsymbol{z} := (z_1, \ldots, z_t)$ be a vector of indeterminates and let $B(\boldsymbol{z})$ denote the block-diagonal matrix with blocks $A_1(\boldsymbol{z}^{\boldsymbol{\mu}_1}), \ldots, A_r(\boldsymbol{z}^{\boldsymbol{\mu}_r})$. Set $T := q^\ell \mathrm{I}_t$. By construction, the functions $g_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i})$, $1 \leq i \leq r$, $1 \leq j \leq m_i$, are related by the T-Mahler system associated with the matrix $B(\boldsymbol{z})$, which proves (d). Since, for every i, $\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$ is regular w.r.t. the q^ℓ-Mahler system associated with the matrix $A_i(z)$, the point $\boldsymbol{\beta}$ is regular w.r.t. the T-Mahler system with matrix $B(\boldsymbol{z})$. Furthermore, since the coordinates of $\boldsymbol{\beta}$ are multiplicatively independent and of modulus smaller that 1, it follows from [2, Theorem 5.9] that $(T, \boldsymbol{\beta})$ is admissible, hence (e) is satisfied. Since $T = q^\ell \mathrm{I}_t$, (f) also holds true. Finally, since the numbers $\alpha_1, \ldots, \alpha_r$ are pairwise multiplicatively independent, so are the numbers $\boldsymbol{\beta}^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{\beta}^{\boldsymbol{\mu}_r}$. Thus, the vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r$ are pairwise linearly independent over $\mathbb{Q}$, which proves (g). ☐

1.3. **Proof of Theorem A.1.** We are now ready to prove our main result. We assume that none of the complex numbers $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ belongs to $\mathbb{K}$, so that it remains to prove that they are algebraically independent over $\overline{\mathbb{Q}}$. We first notice that, according to [1, Corollaire 1.8], this assumption implies that the numbers $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are all transcendental.

Let us divide the natural numbers $1, \ldots, r$ into s classes $\mathcal{I}_1, \ldots, \mathcal{I}_s$, such that i and j belong to the same class if and only if q_i and q_j are multiplicatively dependent. Since an M_q-function is also an M_{q^k}-function for every integer $k \geq 1$, we can assume without any loss of generality that $q_i = q_j := \rho_k$ whenever i and j belong to the same class $\mathcal{I}_k$. Set $\mathcal{E} := \{f_1(\alpha_1), \ldots, f_r(\alpha_r)\}$ and $\mathcal{E}_k := \{f_i(\alpha_i) : i \in \mathcal{I}_k\}$, $1 \leq k \leq s$.

For each $k \in \{1, \ldots, s\}$, we consider the Mahler system given by Proposition A.7 when applied with $q = \rho_k$ and with the pairs $(f_i(z), \alpha_i)$, $i \in \mathcal{I}_k$. Let $\boldsymbol{\beta}_k, (\boldsymbol{\mu}_i)_{i \in \mathcal{I}_k}, T_k, \boldsymbol{z}_k, (g_{i,j}(z))_{i \in \mathcal{I}_k, 1 \leq j \leq m_i}$ and $B_k(\boldsymbol{z}_k)$ denote, respectively, the corresponding algebraic point, vectors of nonnegative integers, transformation, vector of indeterminates, family of M_{ρ_k}-functions and matrix associated with the corresponding T_k-Mahler system. Proposition A.7 ensures that each pair $(T_k, \boldsymbol{\beta}_k)$ is admissible and that the point $\boldsymbol{\beta}_k$ is regular w.r.t. the T_k-Mahler system associated with the matrix $B_k(\boldsymbol{z}_k)$. Since the numbers $\rho_1, \ldots, \rho_s$ are pairwise multiplicatively independent, Condition (f) of Proposition A.7 further implies that the spectral radii of $T_1, \ldots, T_s$ are pairwise multiplicatively independent. Thus, we can apply [2, Corollary 3.9] to these s Mahler systems. We deduce that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{k=1}^{s} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k)\,. \tag{1.5}$$

Now, let us fix $k \in \{1, \ldots, s\}$ and set $\mathcal{F}_k := \{g_{i,j}(\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}) : i \in \mathcal{I}_k,\ 1 \leq j \leq m_i\}$ and

$$\mathcal{F}_{k,i} := \{(g_{i,j}(\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}) : 1 \leq j \leq m_i\}, \quad i \in \mathcal{I}_k\,.$$

Applying [2, Corollary 3.5] to the T_k-Mahler system associated with the matrix $B_k(\boldsymbol{z}_k)$, we obtain that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_k) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_k)}(g_{i,j}(\boldsymbol{z}_k^{\boldsymbol{\mu}_i}) \,:\, i \in \mathcal{I}_k,\, 1 \leq j \leq m_i)\,.$$

Since Condition (g) of Proposition A.7 ensures that the vectors $\boldsymbol{\mu}_i$, $i \in \mathcal{I}_k$, are pairwise linearly independent over $\mathbb{Q}$, it follows from Lemma A.5 that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_k)}(g_{i,j}(\boldsymbol{z}_k^{\boldsymbol{\mu}_i}) \,:\, i \in \mathcal{I}_k,\, 1 \leq j \leq m_i) = \sum_{i \in \mathcal{I}_k} \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}(g_{i,j}(z) \,:\, 1 \leq j \leq m_i)\,.$$

For each $i \in \mathcal{I}_k$, we infer from Condition (c) of Proposition A.7 that we can apply [2, Corollary 3.5] to the Mahler system connecting $g_{i,1}(z), \ldots, g_{i,m_i}(z)$ at the regular point $\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}$. We obtain that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}(g_{i,j}(z) \,:\, 1 \leq j \leq m_i) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_{k,i})\,.$$

Combining these three identities, we get that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_k) = \sum_{i \in \mathcal{I}_k} \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_{k,i})\,. \tag{1.6}$$

We infer from Condition (b) of Proposition A.7 that $f_i(\alpha_i) \in \mathcal{F}_{k,i}$, so that $\mathcal{F}_k = \cup_{i \in \mathcal{I}_k} F_{k,i}$ and $\mathcal{E}_k = \cup_{i \in \mathcal{I}_k} f_i(\alpha_i)$. Then, it follows from [2, Lemma 10.3] and (1.6) that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k) = \sum_{i \in \mathcal{I}_k} \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_i(\alpha_i))\,.$$

Since $f_i(\alpha_i)$ is transcendental for all i, we have $\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_i(\alpha_i)) = 1$ and we deduce that $\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k) = \mathrm{Card}(\mathcal{I}_k)$. Then, it follows from (1.5) that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{k=1}^{s} \mathrm{Card}(\mathcal{I}_k) = r\,.$$

Hence the numbers $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$, just as we wanted. □

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