MAHLER'S METHOD IN SEVERAL VARIABLES AND FINITE AUTOMATA

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Abstract. We develop a theory of linear Mahler systems in several variables from the perspective of transcendence and algebraic independence, which also includes the possibility of dealing with several systems associated with sufficiently independent matrix transformations. Our main results go far beyond the existing literature, also surpassing those of two unpublished preprints the authors made available on the arXiv in 2018. The main new feature is that they apply now without any restriction on the matrices defining the corresponding Mahler systems. As a consequence, we settle several problems concerning expansions of numbers in multiplicatively independent bases. For instance, we prove that no irrational real number can be automatic in two multiplicatively independent integer bases, and we give a new proof and a broad algebraic generalization of Cobham's theorem in automata theory. We also provide a new proof and a multivariate generalization of Nishioka's theorem, a landmark result in Mahler's method.

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1. Introduction

Most of transcendental number theory concerns the study of the algebraic relations over the field of algebraic numbers $\overline{\mathbb{Q}}$ between the values at algebraic

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points of (possibly multivariate) convergent power series with coefficients in $\overline{\mathbb{Q}}$. Given some convergent power series $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$, where $\boldsymbol{z}=(z_1,\ldots,z_n)$, and a point $\boldsymbol{\alpha}\in\overline{\mathbb{Q}}^n$ where these functions are well-defined, one of the main challenges is then to understand to what extent the algebraic relations between $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})$ over the field of rational functions $\overline{\mathbb{Q}}(\boldsymbol{z})$ govern the algebraic relations over $\overline{\mathbb{Q}}$ between the complex numbers $f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})$. In particular, a reoccurring theme consists in establishing the equality

$$\text{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})) = \text{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))\,. \tag{1.1}$$

Equality (1.1) has been notoriously established in the context of Siegel E-functions, leading to the famous Siegel-Shidlovskii theorem [58, Chapter 4], and in that of Mahler M-functions by Ku. Nishioka [49]. More recently (cf. [19, 17, 53, 8, 48]), particular attention has been paid to refining these results, by proving that any homogeneous algebraic relation between the values $f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})$ can be obtained as the specialization at $\boldsymbol{\alpha}$ of a homogeneous algebraic relation between the functions $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})$. These refinements are referred to as *lifting's theorems.*

Let n be a positive integer, $T=(t_{i,j})_{1\le i,j\le n}$ be an $n\times n$ matrix with nonnegative integer coefficients, with which we associate the transformation

$$T\boldsymbol{z} := (z_1^{t_{1,1}}z_2^{t_{1,2}}\cdots z_n^{t_{1,n}},\ldots,z_1^{t_{n,1}}z_2^{t_{n,2}}\cdots z_n^{t_{n,n}})\,,$$

and $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})\in\overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be convergent power series satisfying a linear system of functional equations of the form

$$\begin{pmatrix} f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = A(\boldsymbol{z}) \begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix}, \tag{1.2}$$

where $A(\boldsymbol{z})\in \mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$. Then *Mahler's method* aims at studying the algebraic relations over $\overline{\mathbb{Q}}$ between $f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})$ for suitable algebraic points $\boldsymbol{\alpha}$. In particular, proving that Equality (1.1) holds true under some reasonable assumptions on $A(\boldsymbol{z})$, T, and $\boldsymbol{\alpha}$ is a problem that has its origin in Mahler's pioneering work [39, 40, 41] in the late 1920s.

In this paper, we develop a general theory of linear Mahler systems in several variables from the perspective of transcendental number theory, which also includes the possibility of dealing with several systems associated with sufficiently independent matrix transformations. It is condensed in three main results, Theorems 3.3, 3.6, and 3.8, which go far beyond the existing literature. In particular, Theorem 3.3 is a lifting theorem from which we deduce a multivariate extension of Nishoka's theorem (cf. Corollary 3.5): Equality (1.1) holds true for all $A(\boldsymbol{z})$ under minimal assumptions on T and $\boldsymbol{\alpha}$. These results also surpass those of two unpublished preprints [10, 11] the authors made available on the arXiv in 2018. The main new feature with respect to these two preprints is that our results apply now without any further restriction on the matrices $A(\boldsymbol{z})$.

In this introduction, we focus on the consequence of the results obtained in the multivariate framework for the univariate framework. In the latter, we have $n = 1$, $T = (q)$ for some integer $q \geq 2$, $Tz = z^q$, and the power series involved correspond to the so-called M_q-functions defined by (1.3). Our main result about these functions is Theorem 1.1. In Section 2, we give some applications of Theorem 1.1 to old problems concerning expansions of numbers in multiplicatively independent bases (cf. Theorems 2.2, 2.3, and 2.4). While proving these theorems was our initial objective, it seems to us that the general theory developed in Section 3 to achieve this goal is equally valuable in its own right.

1.1. **Values of M-functions at algebraic points.** Given an integer $q \geq 2$, $f(z) \in \overline{\mathbb{Q}}[[z]]$ is said to be an *M_q-function* or a *q-Mahler function* if there exist polynomials $p_0(z), \ldots, p_m(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_m(z)f(z^{q^m}) = 0. \tag{1.3}$$

We simply say that $f(z)$ is an M-function if it is an M_q-function for some q that we do not need to specify. An M-function is always analytic in some neighborhood of zero and has a meromorphic continuation in the open unit disc. Furthermore, its coefficients generate only a finite field extension of $\mathbb{Q}$. Let us also recall that nonzero complex numbers $x_1, \ldots, x_r$ are multiplicatively independent if there is no nonzero tuple of integers $(n_1, \ldots, n_r)$ such that $x_1^{n_1} \cdots x_r^{n_r} = 1$.

Theorem 1.1. *Let $r \geq 1$ be an integer and $\mathbb{K} \subseteq \overline{\mathbb{Q}}$ be a field. For every integer i, $1 \leq i \leq r$, we let $q_i \geq 2$ be an integer, $f_i(z) \in \mathbb{K}[[z]]$ be an M_{q_i}-function, and $\alpha_i \in \mathbb{K}$, $0 < |\alpha_i| < 1$, be such that $f_i(z)$ is well-defined at α_i. Let us assume that one of the two following properties holds.*

(i) *The numbers $\alpha_1, \ldots, \alpha_r$ are multiplicatively independent.*

(ii) *The numbers $q_1, \ldots, q_r$ are pairwise multiplicatively independent.*

Then $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them belongs to $\mathbb{K}$.

Until now, Theorem 1.1 was only proved when $r = 1$. This special case, conjectured by Cobham [22] in 1968 and settled by the authors [8] in 2017, implies that the decimal expansion of algebraic irrational numbers cannot be generated by finite automata[1].

Remark 1.2. An algorithm to determine whether the numbers $f_i(\alpha_i)$ belong to $\mathbb{K}$ or not is given in [9]. We briefly describe it in Section 11.1.

Let us point out the four main difficulties we have to face when trying to prove Theorem 1.1.

(i) We have to consider a bunch of *arbitrary M-functions*. In contrast, many results in the past were restricted to inhomogeneous order one equations (see Section 3). That is, equations of the form $p_{-1}(z)+p_0(z)f(z)+p_1(z)f(z^q) = 0$. Being able to deal with arbitrary equations becomes essential for applications involving automata.

[1]This result was first proved by Bugeaud and the first author [4] by means of the subspace theorem.

(ii) Given an M-function, we have to consider its values at *arbitrary algebraic points* where it is well-defined, while a classical feature of results in this framework is that they are only available for points which are *regular*[2] with respect to the underlying Mahler system.

(iii) We have to consider simultaneously values of M-functions at *different algebraic points*. In the setting of Siegel E-functions, the study of algebraic relations between values of E-functions at different algebraic points can be achieved by considering different E-functions at the same point. Indeed, if $f(z)$ is an E-function and α is an algebraic number, then the function $f(\alpha z)$ is still an E-function. However, this trick no longer works for M-functions.

(iv) We have to consider M-functions associated with *different transformations* (*i.e.*, $z \mapsto z^q$ with different q).

Thanks to the work of Ku. Nishioka [49], the transcendence theory of linear Mahler systems in one variable is well-developed. It has even reached a rather definitive stage after the recent works of Philippon [53] and the authors [8]. These new results provide tools to overcome (ii), and also (i) in some situations. However, Theorem 1.1 does not fall into the scope of Mahler's method in one variable. In particular, the problem raised by (iii) requires a major development of Mahler's method in several variables. Partial results in this direction are due to Mahler [41], Kubota [33], Loxton and van der Poorten [36, 38], and Nishioka [51]. Last but not least, (iv) is a source of well-known difficulties and only limited results, though of great interest, have been obtained by Nishioka [50] and Masser [44]. *In fine*, the new approach we follow allows us to overcome all the aforementioned difficulties.

1.2. **Organization of the paper.** In Section 2, we first recall a famous conjecture of Furstenberg concerning expansions of real numbers in multiplicatively independent bases. Then we state our main results related to this conjecture, namely Theorems 2.2, 2.3, and 2.4. In Section 3, we state our main results concerning the study of linear Mahler systems in several variables, namely Theorems 3.3, 3.6, and 3.8. We also discuss the three main new ingredients of our approach in Section 3.4. Some notation are introduced in Section 4. As made clear in Section 3, the strength of our results strongly depends on our ability to provide simple and natural conditions that ensure certain admissibility conditions. This problem is addressed in Section 5 where concrete and optimal conditions are given. In Section 6 we prove a new vanishing theorem that is a key ingredient for proving Theorem 3.8. In Section 7, we state Theorem 7.2, a general result dealing with families of linear Mahler systems associated with sufficiently independent transformations. Some preliminary results for proving Theorem 7.2 are gathered in Section 8. Then Theorem 7.2 is proved in Section 9, while Theorems 3.3, 3.6, 3.8, and Corollaries 3.5 and 3.9 are derived from Theorem 7.2 in Section 10. Finally, we deduce Theorem 1.1 from Theorems 3.6 and 3.8 in Section 11, and Theorems 2.2, 2.3, and 2.4 from Theorem 1.1 in Section 12.

[2] See Definition 3.2.

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2. Representing numbers in independent bases

It is commonly expected that expansions of numbers in multiplicatively independent bases, such as 2 and 10, should have no common structure. However, it seems extraordinarily difficult to confirm this naive heuristic principle in some way or another. In the late 1960s, Furstenberg [28, 29] suggested a series of conjectures, which became famous, and aim to capture this heuristic (cf. Conjecture 2.1). Despite recent remarkable progress, Conjecture 2.1 remains totally out of reach of the current methods. As always when mathematicians have to face such an enormous gap between heuristic and knowledge, it becomes essential to find out *good problems*. By that, we mean problems which, on the one hand, formalize and express the general heuristic, and, on the other hand, whose solution does not seem desperately out of reach. While Furstenberg's conjectures take place in a dynamical setting, we use instead the language of automata theory to formulate and solve some related problems that, hopefully, belong to the above category.

2.1. **The dynamical point of view: Furstenberg's conjecture.** The fact that Furstenberg's conjectures take place in a dynamical setting does not come as a great surprise for there is a well-known dictionary transferring combinatorial properties of the expansion of a real number x in an integer base $b \geq 2$ in terms of dynamical properties of the orbit of $\{x\}$ under the map T_b defined on $\mathbb{R}/\mathbb{Z}$ by $x \mapsto bx$. Endowed with the Haar measure, the topological dynamical system $(T_b, \mathbb{R}/\mathbb{Z})$ is ergodic. We let $\mathcal{O}_b(x)$ denote the forward orbit of x under T_b, that is,

$$\mathcal{O}_b(x) := \left\{x, T_b(x), T_b^2(x), \dots\right\} .$$

If $X \subset \mathbb{R}$, we let $\dim_H(X)$ denote the Hausdorff dimension of X and $\overline{X}$ its closure. The entropy of x with respect to the base b is defined as the topological entropy of the dynamical system $(T_b, \overline{\mathcal{O}_b(x)})$. One of Furstenberg's conjecture [29] reads as follows.

Conjecture 2.1 (Furstenberg)**.** *Let b_1 and b_2 be two multiplicatively independent natural numbers, and let $x \in [0,1)$ be a real number. Then*

$$\dim_H \overline{\mathcal{O}_{b_1}(x)} + \dim_H \overline{\mathcal{O}_{b_2}(x)} \geq 1 \,,$$

unless x is rational.

This conjecture has wonderful consequences about expansions of both real and natural numbers. It beautifully expresses the expected balance between the complexity of expansions of an irrational real number in two multiplicatively independent bases:

If x has a simple expansion in base b_1, then it should have a complex expansion in base b_2.

It is easy to see that Conjecture 2.1 holds true generically. Indeed, it follows from the ergodic theorem that

$$\dim_H \overline{\mathcal{O}_{b_1}(x)} = \dim_H \overline{\mathcal{O}_{b_2}(x)} = 1\,,$$

for almost all real numbers x in $[0,1)$. In fact, all the strength of Conjecture 2.1 takes shape when x has a simple expansion in one of the two bases. In particular, Conjecture 2.1 implies that if x has zero entropy in base b_1, then it has a dense orbit under T_{b_2}.

Let us illustrate this with a concrete example. The binary Thue-Morse number τ is defined as follows. Its nth binary digit is equal to 0 if the sum of digits in the binary expansion of n is even, and to 1 otherwise. It is somewhat puzzling that its decimal expansion

$$\langle \tau \rangle_{10} = 0.412\,454\,033\,640\,107\,597\,783\,361\,368\,258\,455\,283\,089\cdots$$

seems unpredictable, while its binary expansion

$$\langle \tau \rangle_2 = 0.011\,010\,011\,001\,011\,010\,010\,110\,011\,010\,011\,001\,011\cdots$$

is, by definition, so regular. This intriguing phenomenon would be nicely explained by Conjecture 2.1. Indeed, since τ has zero entropy in base 2, it should have a dense orbit under T_{10}, meaning that all blocks of digits should occur in its decimal expansion.

Recently, Shmerkin [59] and Wu [62] proved that the set of exceptions to Conjecture 2.1 has Hausdorff dimension zero. Unfortunately, this remarkable results does not tell us anything about expansions of real numbers with zero entropy in some base. Indeed, the set of all such real numbers has Hausdorff dimension zero [45]. Though the works of Shmerkin and Wu mark significant progress towards Conjecture 2.1, the latter remains far out of the reach of current methods. Even worse, we are afraid that their result could be essentially the best dynamical methods have to say about this conjecture.

2.2. **The computational point of view: from finite automata to Mahler's method.** From a computational point of view, there is another relevant notion of simple number, namely the notion of *automatic real number* (see [14, Chapter 13]). While computable numbers can be generated by general Turing machines, automatic numbers are those whose expansion in some base can be generated by a finite automaton. Broadly speaking, a finite automaton is a Turing machine without any memory tape, all its memory being stored in the finite state control. This severe restriction justifies that these numbers are considered as especially simple. For example, the Thue-Morse number τ is automatic in base 2. We refer the reader to [5] and the references therein for a discussion on these different models of computation.

In 1968, Cobham [22] first noticed the following fundamental connection between automatic sequences and M-functions: if the sequence $(a_n)_{n\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ is generated by a finite automaton, then the generating function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

is an M-function. In turn, problems about transcendence and algebraic independence of automatic real numbers can be translated and extended to problems concerning transcendence and algebraic independence of M-functions at algebraic points. Thanks to this connection, the apparently unrelated Theorem 1.1 turns out to be a powerful tool for confirming our general heuristic in this computational setting.

We first deduce from Part (i) of Theorem 1.1 the following result.

Theorem 2.2. *Let b_1 and b_2 be two multiplicatively independent natural numbers. A real number cannot be automatic in both bases b_1 and b_2, unless it is rational.*

This result is a very special case of Conjecture 2.1 since being automatic in some base implies having zero entropy in that base. Nevertheless, proving Theorem 2.2 remained a real challenge. Indeed, until now, for no real number x that is automatic in some base, it had been proved that x could not be automatic in another independent base. We also mention that a weaker version of Theorem 2.2 appears as Open Problems 7 in [14, Chapter 13].

In fact, Part (i) of Theorem 1.1 allows us to deduce the following much stronger result, which has a more Diophantine flavor.

Theorem 2.3. *Let $r \geq 1$ be an integer. Let $b_1, \ldots, b_r$ be multiplicatively independent positive integers, and, for every i, $1 \leq i \leq r$, let x_i be a real number that is automatic in base b_i. Then the numbers $x_1, \ldots, x_r$ are algebraically independent over $\overline{\mathbb{Q}}$, unless one of them is rational.*

Unlike Theorem 2.2, Theorem 2.3 is not implied by Furstenberg's conjecture. Theorem 2.3 does not only imply that the Thue-Morse number τ cannot be automatic in base 10, but also that this is the case for any number obtained from τ by using algebraic numbers and algebraic operations (addition, multiplication, division, taking nth roots...). The case $r = 1$ was a long-standing conjecture first proved by Bugeaud and the first author [4] by means of the subspace theorem. See also [8, 53] for a recent different proof based on Mahler's method. So far, Theorem 2.3 was only known in that particular case.

2.3. **Representing natural numbers in independent bases.** Other astonishing consequences of Furstenberg's conjecture concern expansions of natural numbers. For instance, using an elementary construction, Furstenberg shown in [29] how to deduce from Conjecture 2.1 that any finite block of digits occurs in the decimal expansion of 2^n, as soon as n is large enough. Note that, in the same vein, a conjecture of Erdös claims that the digit 2 occurs in the ternary expansion of 2^n for all $n > 8$ (see, for instance, [26]). These kinds of problems are notoriously difficult.

Theorem 1.1 has also valuable consequences about expansions of natural numbers. We recall that a set $\mathcal{E} \subset \mathbb{N}$ is q-automatic if its elements, when written in base q, can be recognized by a finite automaton (cf. [14, Chapter 5]). In that case, the generating series $\sum_{n\in\mathcal{E}} z^n$ is an M_q-function. In this framework, Cobham [23] proved the following famous theorem: a set $\mathcal{E} \subset \mathbb{N}$ that is both p- and q-automatic, where p and q are multiplicatively independent, is necessarily periodic (*i.e.*, the union of a finite set and finitely many arithmetic progressions). Cobham's theorem can be rephrased in terms of power series as follows: if $\mathcal{E}$ is both p- and q-automatic, then its generating series is a rational function.

In 1987, Loxton and van der Poorten [55] conjectured the following generalization: if a power series is both an M_p-function and an M_q-function then it is rational. This conjecture was first proved by Bell and the first author in [2], while a different proof was given by Schäfke and Singer [57]. Very recently, the authors of [7] even proved a stronger result also conjectured by Loxton and van der Poorten [55]: given any M_p-function $f(z)$ and any M_q-function $g(z)$, then $f(z)$ and $g(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, unless one of them is rational. This result refines Cobham's theorem by expressing, in algebraic terms, the discrepancy between aperiodic automatic sets associated with multiplicatively independent input bases. The proof given in [7] is based on a suitable parametrized Galois theory associated with linear Mahler equations and follows the strategy initiated in [6].

Part (ii) of Theorem 1.1 leads to the following significant generalization of all the aforementioned results, providing in particular a totally new proof of Cobham's theorem.

Theorem 2.4. *Let $r \geq 1$ be an integer. For every integer i, $1 \leq i \leq r$, let $q_i \geq 2$ be an integer and $f_i(z) \in \overline{\mathbb{Q}}\{z\}$ be an M_{q_i}-function. Assume that $q_1, \ldots, q_r$ are pairwise multiplicatively independent. Then $f_1(z), \ldots, f_r(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, unless one of them is rational.*

The case $r = 1$ is well-known (cf. [52, Theorem 5.1.7]), while, as previously mentioned, the case $r = 2$ is much harder and was only recently proved in [7]. So far, Theorem 2.4 was only proved in these particular cases.

Finally, let us mention that similar results have been obtained by Zannier [63] and, more recently, by Medvedev, Nguyen, and Scanlon [46] for solutions of some nonlinear Mahler equations of order one.

3. Mahler's method in several variables

In this section, we state our main results concerning the study of linear Mahler systems in several variables form the perspective of transcendental number theory.

3.1. **Mahler's transformations and linear Mahler systems.**

Let n be a positive integer and $T = (t_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with nonnegative integer coefficients. Given a n-tuple of indeterminates $\boldsymbol{z} = (z_1, \ldots, z_n)$, we set

$$T\boldsymbol{z} := (z_1^{t_{1,1}} z_2^{t_{1,2}} \cdots z_n^{t_{1,n}}, \ldots, z_1^{t_{n,1}} z_2^{t_{n,2}} \cdots z_n^{t_{n,n}}),$$

and we let also T act on $\mathbb{C}^n$ in a similar way. We let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers which embeds into the field $\mathbb{C}$ of complex numbers, and $\overline{\mathbb{Q}}^\star := \overline{\mathbb{Q}} \setminus \{0\}$. If $\mathbb{K}$ is a subfield of $\mathbb{C}$, we let $\mathbb{K}\{\boldsymbol{z}\}$ denote the ring of formal power series with coefficients in $\mathbb{K}$ that converges in some neighborhood of the origin.

Definition 3.1. A *linear T-Mahler system*, or simply a *Mahler system*, is a system of functional equations of the form

$$\begin{pmatrix} f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = A(\boldsymbol{z}) \begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix}, \tag{3.1}$$

where $A(\boldsymbol{z}) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$. A multivariate convergent power series $f(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ is said to be a *Mahler function* if it is a coordinate of a vector representing a solution to a linear Mahler system.

Definition 3.2. A point $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is said to be *regular* with respect to the Mahler system (3.1) if the matrix $A(\boldsymbol{z})$ is well-defined and invertible at $T^k\boldsymbol{\alpha}$ for all nonnegative integers k.

Warning. Independently of the choice of the Mahler system (3.1), there are some unavoidable restrictions that one has to impose on the matrix transformation T and on the algebraic point $\boldsymbol{\alpha}$. When these conditions are fulfilled, the pair $(T, \boldsymbol{\alpha})$ is said to be *admissible*. We postpone the definition of an admissible pair to Section 5, but let us just already say that, in this respect, our results are as general as possible. With this formalism, all our results are concerned with values at some algebraic point $\boldsymbol{\alpha}$ of some Mahler functions $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ related by a system of the form (3.1) under the assumption that:

(a) the pair $(T, \boldsymbol{\alpha})$ is admissible,
(b) $\boldsymbol{\alpha}$ is regular with respect to (3.1).

As discussed in [1], these conditions are typical in Mahler's method.

3.2. **The lifting theorem.** As a first contribution, we prove the following result. Let us recall that a field extension $\mathbb{L}$ of a field $\mathbb{K}$ is said to be *regular*[3] if $\mathbb{K}$ is algebraically closed in $\mathbb{L}$ and $\mathbb{L}$ is separable over $\mathbb{K}$. If $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^n$, we let $\overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$ denote the algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ in $\overline{\mathbb{Q}}\{\boldsymbol{z} - \boldsymbol{\alpha}\}$.

Theorem 3.3 (Lifting). *Let $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be related by a system of functional equations of the form (3.1). Let us assume that $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is a regular point with respect to (3.1) and that the pair $(T, \boldsymbol{\alpha})$ is admissible. Then for every homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that*

$$P(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) = 0\,,$$

there exists a homogeneous polynomial $Q \in \overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}[X_1, \ldots, X_m]$ such that

$$Q(\boldsymbol{z}, f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})) = 0 \quad \text{and} \quad Q(\boldsymbol{\alpha}, X_1, \ldots, X_m) = P(X_1, \ldots, X_m).$$

Furthermore, if $\overline{\mathbb{Q}}(\boldsymbol{z})(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$, then there exists such a polynomial Q in $\overline{\mathbb{Q}}[\boldsymbol{z}, X_1, \ldots, X_m]$.

Theorem 3.3 is the first result in this area that applies to *all* linear Mahler systems in several variables, that is, without any restriction on the matrix $A(\boldsymbol{z})$.

Remark 3.4. Theorems 3.3 also applies to inhomogeneous polynomial relations, for we can always turn an inhomogeneous relation into a homogeneous one by adding the constant function $f_0 \equiv 1$ to the system and replacing the matrix $A(\boldsymbol{z})$ by

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A(\boldsymbol{z}) \end{array}\right).$$

[3]The reader will take care that we use two totally different notions of regularity in this paper.

As a corollary of the lifting theorem, we deduce the following multivariate extension of Nishioka's theorem.

Corollary 3.5. *Let* $f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})\in\overline{\mathbb{Q}}\{\boldsymbol{z}\}$ *be related by a system of the form* (3.1)*. Let us assume furthermore that* $\boldsymbol{\alpha}\in(\overline{\mathbb{Q}}^\star)^n$ *is a regular point with respect to* (3.1) *and that the pair* $(T,\boldsymbol{\alpha})$ *is admissible. Then*

$$\text{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha}))=\text{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))\,. \tag{3.2}$$

Now, let us compare Theorem 3.3 and Corollary 3.5 with previous results on the subject.

The case $n=1$. In that case, the operator T takes the simple form $z\mapsto z^q$, where $q\geq 2$ is an integer, and the pair (T,α) is admissible as soon as $0<|\alpha|<1$. Furthermore, the field extension $\overline{\mathbb{Q}}(z)(f_1(z),\ldots,f_m(z))$ is always regular. After several partial results due to Mahler, Kubota, and Loxton and van der Poorten, Ku. Nishioka [49] finally proved in 1990 that

$$\text{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha),\ldots,f_m(\alpha))=\text{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\ldots,f_m(z))$$

for all matrices $A(z)$ and all regular points $\alpha\in\overline{\mathbb{Q}}$, $0<|\alpha|<1$. This is certainly a landmark result in Mahler's method, which corresponds to the case $n=1$ of Corollary 3.5. The proof of Nishioka's theorem is based on some technics from commutative algebra first introduced in the framework of algebraic independence by Nesterenko. More recently, Philippon [53] and then the authors [8] have refined Nishioka's theorem by proving the case $n=1$ of Theorem 3.3, which we refer to as Philippon's lifting theorem. Similar lifting theorems have first been obtained in the framework of Siegel E-functions by Nesterenko and Shidlovskii [47], by Beukers [19] using some results of André [15, 16] on the theory of E-operators, and then by André [17]. A recent proof of Philippon's lifting theorem in the spirit of [17] is also given in [48]. In [53, 8, 48], the latter is derived from Nishioka's theorem, while our proof of Theorem 3.3 has little in common with these papers and [49]. In particular, it provides a new and elementary way to prove the theorems of Nishioka and Philippon, which we have detailed in the subsequent paper [12].

The case $n\geq 2$. Unfortunately, the method used in [49] for proving Nishioka's theorem hardly generalizes to higher dimension. In 1982, Loxton and van der Poorten [38] published a paper claiming that Equality (3.2) holds true when the matrix $A(\mathbf{0})$ is well-defined and nonsingular, the pair $(T,\boldsymbol{\alpha})$ is admissible, and $\boldsymbol{\alpha}$ is a regular algebraic point. It was the main result published in this area, but unfortunately some argument in their proof is flawed. This is reported, for instance, by Nishioka in [49]. In the end, Mahler's method in several variables has only been successfully applied to the following two much more restricted classes of matrices. In 1977, Kubota [33] proved that Equality (3.2) holds true when the matrix $A(\boldsymbol{z})$ is *almost diagonal*, that is, when the functions $f_i(\boldsymbol{z})$ satisfy a system of equations of the form

$$\begin{pmatrix} 1 \\ f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 \cdots & & 0 \\ \hline b_1(\boldsymbol{z}) & a_1(\boldsymbol{z}) & & \\ \vdots & & \ddots & \\ b_m(\boldsymbol{z}) & & & a_m(\boldsymbol{z}) \end{array}\right) \begin{pmatrix} 1 \\ f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix}$$

where $a_i(\boldsymbol{z}), b_i(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$ have no pole at $\mathbf{0}$, and $a_i(\mathbf{0}) \neq 0$. Such systems are precisely those arising from the study of several inhomogeneous equations of order one. A variant of this result is due to Nishioka [51], who proved in 1996 that Equality (3.2) also holds true when the matrix $A(\boldsymbol{z})$ is *almost constant*, that is, for systems of the form

$$\begin{pmatrix} 1 \\ f_1(T\boldsymbol{z}) \\ \vdots \\ f_m(T\boldsymbol{z}) \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline b_1(\boldsymbol{z}) & & & \\ \vdots & & B & \\ b_m(\boldsymbol{z}) & & & \end{array}\right) \begin{pmatrix} 1 \\ f_1(\boldsymbol{z}) \\ \vdots \\ f_m(\boldsymbol{z}) \end{pmatrix}$$

where $B \in \mathrm{GL}_m(\overline{\mathbb{Q}})$, and $b_i(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$ have no pole at $\mathbf{0}$. The proof of these results (and also of the failed attempt by Loxton and van der Poorten) follow closely the approach initiated by Mahler in [41]. We stress that, so far, this remained the only available strategy to tackle this problem (see Section 3.4).

3.3. **The two purity theorems.** According to the lifting theorem, the study of the algebraic relations between the values of Mahler functions related by a system of equations of the form (3.1) can be reduced to the easier study of the algebraic relations between the functions themselves. However, easier does not necessarily mean *easy*, and, so far, only the linear relations between M-functions have been fully understood [8, 9]. Our second main result is of a different nature. It states that, when evaluated at sufficiently independent algebraic points, Mahler functions associated with transformations having the same spectral radius *always* behave independently. The main feature of this result is that there is no need to check any kind of independence between the Mahler functions under consideration.

To state this result properly, we first need some notation. Let us consider several tuples of complex numbers

$$\mathcal{E}_1 = (\zeta_{1,1}, \ldots, \zeta_{1,s_1}), \ldots, \mathcal{E}_r = (\zeta_{r,1}, \ldots, \zeta_{r,s_r}) \,.$$

With every i, $1 \leq i \leq r$, we associate a vector of indeterminates $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,s_i})$, and we let

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i) := \left\{ P(\boldsymbol{X}_i) \in \overline{\mathbb{Q}}[\boldsymbol{X}_i] : P(\zeta_{i,1}, \ldots, \zeta_{i,s_i}) = 0 \right\}$$

denote the ideal of algebraic relations over $\overline{\mathbb{Q}}$ between the coordinates of $\mathcal{E}_i$. We also consider the tuple $\mathcal{E} = (\zeta_{1,1}, \ldots, \zeta_{r,s_r})$ obtained by concatenation of the tuples $\mathcal{E}_i$, and we set $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$ and

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) := \left\{ P(\boldsymbol{X}) \in \overline{\mathbb{Q}}[\boldsymbol{X}] : P(\zeta_{1,1}, \ldots, \zeta_{r,s_r}) = 0 \right\} \,.$$

We say that $P \in \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ is a *pure algebraic relation* with respect to $\mathcal{E}_i$ if it belongs to the extended ideal

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) := \mathrm{span}_{\overline{\mathbb{Q}}[\boldsymbol{X}]}\{P(\boldsymbol{X}_i) : P \in \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\} \,.$$

Our second main result reads as follows.

Theorem 3.6 (Purity–Independent points)**.** *Let $r \geq 2$ be an integer. For every integer i, $1 \leq i \leq r$, let us consider a linear Mahler system*

$$(3.3.i) \qquad \begin{pmatrix} f_{i,1}(T_i \boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(T_i \boldsymbol{z}_i) \end{pmatrix} = A_i(\boldsymbol{z}_i) \begin{pmatrix} f_{i,1}(\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(\boldsymbol{z}_i) \end{pmatrix}$$

where $A_i(\boldsymbol{z}_i)$ belongs to $\mathrm{GL}_{m_i}(\overline{\mathbb{Q}}(\boldsymbol{z}_i))$, $\boldsymbol{z}_i := (z_{i,1}, \ldots, z_{i,n_i})$ is a tuple of indeterminates, and T_i is an $n_i \times n_i$ matrix with nonnegative integer coefficients and with spectral radius $\rho(T_i)$. Let $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,n_i}) \in (\overline{\mathbb{Q}}^\star)^{n_i}$, $\mathcal{E}_i$ be a subtuple of $(f_{i,1}(\boldsymbol{\alpha}_i), \ldots, f_{i,m_i}(\boldsymbol{\alpha}_i))$, and $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_r)$. Suppose that the two following conditions hold.

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t. (3.3.i) and $(T_i, \boldsymbol{\alpha}_i)$ is admissible.*
(ii) *$\rho(T_1) = \cdots = \rho(T_r)$ and there is no nonzero tuple $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_r) \in \mathbb{Z}^N$, $N = n_1 + \cdots + n_r$, such that $(T_1^k \boldsymbol{\alpha}_1)^{\boldsymbol{\mu}_1} \cdots (T_r^k \boldsymbol{\alpha}_r)^{\boldsymbol{\mu}_r} = 1$, for all k in an infinite arithmetic progression.*

Then

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \,.$$

In other words, the only algebraic relations between the coordinates of $\mathcal{E}$ are those that can be trivially derived from the pure algebraic relations with respect to the coordinates of each $\mathcal{E}_i$.

The first results dealing with values of Mahler functions at independent points are due to Mahler [40] and are limited to linear independence over $\overline{\mathbb{Q}}$. Some generalization are due to Kubota [33] and to Loxton and van der Poorten [36]. All these results are restricted to the study of several inhomogeneous equations of order one.

Remark 3.7. Condition (ii) is clearly satisfied when all the algebraic numbers $\alpha_{1,1}, \ldots, \alpha_{r,n_r}$ are multiplicatively independent.

Let us turn to our third main result. It states that values at algebraic points of Mahler functions associated with sufficiently independent transformations *always* behave independently. As with Theorem 3.6, the main advantage is that there is no need to check any kind of functional independence. Again, this result is expressed in terms of purity.

Theorem 3.8 (Purity–Independent transformations)**.** *We continue with the notation of Theorem 3.6. Suppose that the two following conditions hold.*

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t. (3.3.i) and $(T_i, \boldsymbol{\alpha}_i)$ is admissible.*
(ii) *The spectral radii $\rho(T_1), \ldots, \rho(T_r)$ are pairwise multiplicatively independent.*

Then

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \,.$$

In 1976, Kubota [32] and van der Poorten [54], first envisaged the possibility of extending Mahler's method in order to consider simultaneously several

Mahler systems associated with independent transformations. In [32], Kubota gave a sketch of proof in a very specific case and announced a paper on this problem, but the latter never appeared in print. Then Loxton and van der Poorten [37] stated some related result, but the corresponding proof is incomplete (see [50, p. 89]). In 1987, van der Poorten [55] made this guess more ambitious and precise, pointing out several striking consequences that would follow from results he expected to prove in his collaboration with Loxton. However, these authors did not publish any new paper on this problem. In the end, only examples limited to the study of several inhomogeneous equations of order one have been worked out by Nishioka [50] and Masser [44]. In contrast, Theorem 3.8 applies to arbitrary linear Mahler systems, and to a much larger class of transformation matrices and algebraic points.

Of course, Theorems 3.6 and 3.8 are strong statements about algebraic independence.

Corollary 3.9. *We continue with the assumptions of Theorems 3.6 or 3.8. The following equality holds true:*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,.$$

Remark 3.10. In geometric terms, Theorems 3.6 and 3.8 can be rephrased by saying that the affine $\overline{\mathbb{Q}}$-variety associated with the ideal $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ is isomorphic to the cartesian product of the affine $\overline{\mathbb{Q}}$-varieties associated with the ideals $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$, $1 \leq i \leq r$. Indeed, we prove that their coordinate rings are isomorphic. That is,

$$\frac{\overline{\mathbb{Q}}[\boldsymbol{X}]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})} \cong \frac{\overline{\mathbb{Q}}[\boldsymbol{X}_1]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_1)} \otimes_{\overline{\mathbb{Q}}} \cdots \otimes_{\overline{\mathbb{Q}}} \frac{\overline{\mathbb{Q}}[\boldsymbol{X}_r]}{\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_r)}\,.$$

3.4. **Main new ingredients.** As already mentioned, all previous results concerning the transcendence theory of linear Mahler systems in several variables are very much inspired by the early work of Mahler [41]. We also start with the same initial strategy, but we add a number of fundamental new ingredients, including Hilbert's Nullstellensatz, tools from ergodic Ramsey theory, and a new vanishing theorem.

In all previous works, a crucial step consists in expressing the coordinates of the iterated matrix $A_k(\boldsymbol{z}) := A(\boldsymbol{z})A(T\boldsymbol{z})\cdots A(T^{k-1}\boldsymbol{z})$ associated with the Mahler system (3.1) in terms of linear combinations of some convergent power series of the form $g_i(T^k\boldsymbol{z})$, possibly twisted by some multivariate exponential polynomials. This is really of great importance for one can then apply some vanishing theorems to the power series $g_i(\boldsymbol{z})$. This step has gradually become more difficult in the aforementioned works, as the matrices under consideration have taken a more general form. Its complexity culminated in [10]. Unfortunately, one cannot expect to find this kind of expression when $A(\boldsymbol{z})$ is not regular singular in the sense of [10, Definition 1.1]. Hence this strategy suffers from an intrinsic limitation, which prevents it from dealing with arbitrary Mahler systems. We overcome this main deficiency by defining the so-called *relation matrices* in Section 8. Their existence and main properties are obtained by means of Hibert's Nullstellensatz and the notion of piecewise syndetic set. Introducing these matrices is a cornerstone of the present work and the main novelty with respect to our two unpublished preprints [10, 11].

In order to apply our results to transformations T and points $\boldsymbol{\alpha}$ that are as general as possible, it is of great importance to prove suitable *vanishing theorems*. That is, results that guarantee the nonvanishing of arbitrary multivariate analytic functions at special sets of points (typically, certain subsets of $\{T^k\boldsymbol{\alpha},\, k \geq 0\}$). In the case of a single transformation, Masser [43] solved this problem in a rather definitive way. We note that Masser's vanishing theorem (in fact a refinement using the notion of piecewise syndetic set) is already strong enough to prove Theorems 3.3 and 3.6. In fact, as shown in [12], we only need the identity theorem for reproving Nishioka's theorem and Philippon's lifting theorem. Unfortunately, Masser's vanishing theorem is not suited to deal with Mahler systems associated with independent transformations. First results towards this goal were proved by Nishioka [50] and, again, by Masser [44]. Unfortunately, they remain too restricted for proving Theorem 3.8. In 2005, Corvaja and Zannier [24, Theorem 3] deduced from the subspace theorem a general result concerning the vanishing at $\mathcal{S}$-units of analytic multivariate power series with algebraic coefficients. They already noticed that it could be relevant for Mahler's method. Using the flexibility of their result and the notion of piecewise syndetic set, we cook up in Section 6 our own vanishing theorem, which is specifically shaped for our purpose.

3.5. **Relevance of Mahler's formalism.** To end this section, we recall two major advantages that this multivariate formalism offers.

First, adding variables makes it possible to deal with values of M-functions at *different algebraic points*. Let us give a basic example. With the function $\mathfrak{f}(z) = \sum_{n=0}^{\infty} z^{2^n}$, we can associate the two variables linear T-Mahler system

$$(3.4)\qquad \begin{pmatrix} 1 \\ \mathfrak{f}(z_1^2) \\ \mathfrak{f}(z_2^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -z_1 & 1 & 0 \\ -z_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{f}(z_1) \\ \mathfrak{f}(z_2) \end{pmatrix}, \quad \text{where } T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

By Theorem 5.10, the point $\boldsymbol{\alpha} := (1/2, 1/3)$ is regular w.r.t. (3.4) and the pair $(T, \boldsymbol{\alpha})$ is admissible. The key point is that the transcendence of $\mathfrak{f}(z)$ gives *for free* the algebraic independence over $\overline{\mathbb{Q}}(z_1, z_2)$ of the functions $\mathfrak{f}(z_1)$ and $\mathfrak{f}(z_2)$. By Corollary 3.5, it follows that $\mathfrak{f}(1/2)$ and $\mathfrak{f}(1/3)$ are algebraically independent over $\overline{\mathbb{Q}}$. This important principle really takes shape, and acquires great generality, with Theorems 1.1 and 3.6.

The second advantage of Mahler's multivariate formalism comes from the possibility of dealing with a much larger class of one-variable functions obtained by suitable specializations of Mahler functions in several variables. Mahler's favorite example was the family of the Hecke-Mahler functions

$$\mathfrak{f}_\omega(z) = \sum_{n=0}^{\infty} \lfloor n\omega \rfloor z^n ,$$

where ω is a quadratic irrational real number. Though $\mathfrak{f}_\omega(z)$ is not an M-function[4], we have that $\mathfrak{f}_\omega(z) = F_\omega(z,1)$, where

$$F_\omega(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\lfloor n_1\omega \rfloor} z_1^{n_1} z_2^{n_2}$$

is a Mahler function in two variables. In another direction, Cobham [22] proved that generating functions of morphic sequences are specializations of the form $F(z,\ldots,z)$ for some multivariate Mahler functions $F(z_1,\ldots,z_n)$. Some related applications of our main results can be found in [11, 13].

4. Notation

We fix here some notation that we will use all along this paper. We let $\mathbb{N} := \{0,1,2,\ldots\}$ denote the set of nonnegative integers. Given a field $\mathbb{K}$, we let $\mathbb{K}^\star$ denote the set $\mathbb{K}\setminus\{0\}$. Given a field extension $\mathbb{L}$ of $\mathbb{K}$, and elements $a_1,\ldots,a_m$ in $\mathbb{L}$, we let $\mathrm{tr.deg}_{\mathbb{K}}(a_1,\ldots,a_m)$ denote the transcendence degree over $\mathbb{K}$ of the field extension $\mathbb{K}(a_1,\ldots,a_m)$.

4.1. **Matrices and vectors.** We draw the reader's attention to the fact that, all along this paper, we let matrices act on vectors in several different ways.

4.1.1. *Nonlinear action of matrices with nonnegative integer coefficients.* Let d be a positive integer and $\mathcal{M}_d(\mathbb{N})$ denote the set of $d\times d$ matrices with nonnegative integer coefficients. The functional equations considered in Mahler's method involve a nonusual action of such matrices on vectors of complex numbers (usually of algebraic numbers), as well as on vectors of indeterminates. If $T = (t_{i,j})_{1\le i,j\le d} \in \mathcal{M}_d(\mathbb{N})$ and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d$, we set

$$T\boldsymbol{\alpha} := (\alpha_1^{t_{1,1}}\alpha_2^{t_{1,2}}\cdots\alpha_d^{t_{1,d}},\ldots,\alpha_1^{t_{d,1}}\alpha_2^{t_{d,2}}\cdots\alpha_d^{t_{d,d}})\,.$$

Similarly, given a d-tuple of indeterminates $\boldsymbol{z} = (z_1,\ldots,z_d)$, we set

$$T\boldsymbol{z} := (z_1^{t_{1,1}}z_2^{t_{1,2}}\cdots z_d^{t_{1,d}},\ldots,z_1^{t_{d,1}}z_2^{t_{d,2}}\cdots z_d^{t_{d,d}})\,.$$

This action is generally not linear. In order to limit confusion as much as possible, we will always use the letter T, adding possibly some subscripts (*i.e.*, using the general form T_*), when using this specific action.

4.1.2. *Linear action of general matrices.* For matrices which are not of the previous sort (*i.e.*, not of the form T_*), we keep the standard notation for products of matrices and vectors. Let n,m,r be positive integers and $\mathbb{K}$ be a field. Given $A\in\mathcal{M}_{n\times m}(\mathbb{K})$ and $B\in\mathcal{M}_{m\times r}(\mathbb{K})$, we write $AB\in\mathcal{M}_{n\times r}(\mathbb{K})$ for the usual linear action of matrices. In particular, if $A\in\mathcal{M}_{n\times m}(\mathbb{K})$ and $\boldsymbol{x}\in\mathbb{K}^m$ is a column vector (resp. $\boldsymbol{x}\in\mathbb{K}^n$ a row vector), we write $A\boldsymbol{x}$ (resp. $\boldsymbol{x}A$) for the usual matrix product between A and $\boldsymbol{x}$ (resp. $\boldsymbol{x}$ and A). We also let ${}^{\mathrm{t}}A\in\mathcal{M}_{m\times n}(\mathbb{K})$ denote the transpose of A. If $A_1,\ldots,A_r$ are matrices

[4]This fact only very recently became known. It is an easy consequence of Theorem 3.8, but it can also be obtained by combining the results in [3] and [25].

with coefficients in $\mathbb{K}$, we let $A_1 \oplus \cdots \oplus A_r$ denote the direct sum of these matrices. That is,

$$A_1 \oplus \cdots \oplus A_r := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix} .$$

We let $\mathcal{M}_n(\mathbb{K})$ denote the set of $n \times n$ matrices with coefficients in $\mathbb{K}$ and I_n denote the identity matrix of size n. If $A \in \mathcal{M}_n(\mathbb{C})$, we let $\rho(A)$ denote its spectral radius, that is the maximum of the modulus of its eigenvalues.

4.1.3. *Warning.* There are few places where we have to use the classical linear action for some matrices of the form $T_* \in \mathcal{M}_d(\mathbb{N})$. First, the right action of T_* is always linear, that is if $\boldsymbol{\lambda} \in \mathbb{Z}^d$ is a row vector, we let $\boldsymbol{\lambda} T_*$ denote the usual matrix product between $\boldsymbol{\lambda}$ and T_*. Secondly, in Sections 5 and 6, there are few places where we have to consider the usual matrix product between a matrix $T_* \in \mathcal{M}_d(\mathbb{N})$ and a vector $\boldsymbol{x} \in \mathbb{C}^d$. There we write $T_*(\boldsymbol{x})$, adding parentheses, in order to make a clear distinction with the nonlinear action of T_* as in $T_*\boldsymbol{\alpha}$.

4.1.4. *Multidimensional powers.* Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$ be two vectors in $\mathbb{N}^d$. We define a partial order on $\mathbb{N}^d$ by setting $\boldsymbol{\lambda} \leq \boldsymbol{\gamma}$ if $\lambda_i \leq \gamma_i$, $1 \leq i \leq d$. Similarly, we define a strict partial order on $\mathbb{N}^d$ by setting $\boldsymbol{\lambda} < \boldsymbol{\gamma}$ if $\lambda_i < \gamma_i$, $1 \leq i \leq d$. When $\boldsymbol{\lambda} \leq \boldsymbol{\gamma}$ we also set

$$\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} := \prod_{i=1}^{d} \binom{\gamma_i}{\lambda_i} ,$$

the product of binomial coefficients associated with each coordinate of $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. Given two positive integers d, h, a matrix $M = (a_{i,j}) \in \mathcal{M}_{d,h}(R)$ with coefficients in some ring R, and a matrix $\boldsymbol{\mu} = (\mu_{i,j}) \in \mathcal{M}_{d,h}(\mathbb{Z})$, we set

$$M^{\boldsymbol{\mu}} := \prod_{i=1}^{d} \prod_{j=1}^{h} a_{i,j}^{\mu_{i,j}} ,$$

if the product exists. In particular, if $h = 1$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\overline{\mathbb{Q}}^\star)^d$, then $\boldsymbol{\alpha}^{\boldsymbol{\mu}}$ stands for $\alpha_1^{\mu_1} \cdots \alpha_d^{\mu_d}$. Thus, if $T \in \mathcal{M}_d(\mathbb{N})$, we get that $(T\boldsymbol{\alpha})^{\boldsymbol{\mu}} = \boldsymbol{\alpha}^{\boldsymbol{\mu}T}$, where as previously mentioned $T\boldsymbol{\alpha}$ refers to the nonlinear left action of T, while $\boldsymbol{\mu}T$ refers to the usual linear right action of T.

4.2. **Norms.** We let $|\cdot|$ denote the 1-norm on $\mathbb{C}^d$, that is $|\boldsymbol{x}| := |x_1| + \cdots + |x_d|$ for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{C}^d$, where $|x_i|$ stands for the modulus of x_i. In particular, when $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d$, we have $|\boldsymbol{\lambda}| = \lambda_1 + \cdots + \lambda_d$. We also let $\| \cdot \|$ denote the maximum norm of matrices with complex numbers, that is the maximum of the modulus of their coefficients. This will be used for square matrices, row vectors, and column vectors.

4.3. **Asymptotics.** We use the standard Landau notation $\mathcal{O}$, as well as the usual equivalence relation $\sim$. We also use the notation $\gg$ as follows. Let $\lambda_1, \ldots, \lambda_n$ be some integer parameters. Writing that some property holds true for all $\lambda_1 \gg 1$ means that it holds true for all λ_1 large enough, while writing that it holds true for all $\lambda_1 \gg \lambda_2, \ldots, \lambda_n$ means that it holds true for all λ_1 that is sufficiently large w.r.t. all the parameters $\lambda_2 \ldots, \lambda_n$. Finally, writing that some property holds true for all $\lambda_1 \gg \lambda_2 \gg \lambda_3$ means that it holds true for all λ_1 that is sufficiently large w.r.t. λ_2, assuming that λ_2 is itself sufficiently large w.r.t. λ_3.

4.4. **Convergent power series.** Given a positive real number R and $\boldsymbol{\alpha} \in \mathbb{C}^d$, we let

$$\mathcal{D}(\boldsymbol{\alpha}, R) := \{\boldsymbol{\theta} \in \mathbb{C}^d \; : \; \|\boldsymbol{\theta} - \boldsymbol{\alpha}\| < R\}$$

denote the open polydisc with center $\boldsymbol{\alpha}$ and radius R. By definition, an element $g \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ has a unique expansion of the form

$$g(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^d} g_{\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} ,$$

which converges in some neighborhood of the origin. The *radius of convergence* of g is defined as the supremum of the positive real numbers R such that the power series defining g is convergent on $\mathcal{D}(\mathbf{0}, R)$. By [21, Proposition 2.2], when the radius of convergence of g is finite and equal to R_0, the power series defining g is absolutely convergent on $\mathcal{D}(\mathbf{0}, R_0)$. By specialization, we deduce from the Cauchy-Hadamard theorem that if $g(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^d} g_{\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$, then

$$|g_{\boldsymbol{\lambda}}| = \mathcal{O}\left(R^{-|\boldsymbol{\lambda}|}\right) , \tag{4.1}$$

for all positive real numbers R smaller than the radius of convergence of g.

4.5. **Height.** We will only need some basic properties of the absolute Weil height and we refer the interested reader to [61, Chapter 3] for more details. Let $\mathbb{K}$ be a number field, $[\mathbb{K} : \mathbb{Q}]$ its degree, and $\mathsf{M}_{\mathbb{K}}$ be the set of places of $\mathbb{K}$. With each place $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, we can associate a normalized absolute value $| \cdot |_{\mathsf{v}}$ and a positive integer d_{v} such that the product formula holds:

$$\prod_{\mathsf{v} \in \mathsf{M}_{\mathbb{K}}} |\alpha|_{\mathsf{v}}^{d_{\mathsf{v}}} = 1 , \quad \forall \alpha \in \mathbb{K}^\star .$$

Furthermore, this normalization ensures that the restriction of $| \cdot |_{\mathsf{v}}$ to $\mathbb{Q}$ corresponds to one of the classical absolute value of $\mathbb{Q}$ (*i.e.*, the classical Archimedean absolute value or a p-adic absolute value normalized such that $|p|_p = 1/p$). The *absolute Weil height* of a point $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{K}^d$ is then defined by

$$H(\boldsymbol{\beta}) := \prod_{\mathsf{v} \in \mathsf{M}_{\mathbb{K}}} \max\left(1, |\beta_1|_{\mathsf{v}}, \ldots, |\beta_d|_{\mathsf{v}}\right)^{d_{\mathsf{v}}/[\mathbb{K}:\mathbb{Q}]} .$$

The value $H(\boldsymbol{\beta})$ in this definition does not depend on the choice of a number field $\mathbb{K}$ containing $\beta_1, \ldots, \beta_d$. When $d = 1$ and $\beta \in \mathbb{K}$, we obtain that

$$H(\beta) = \prod_{\mathsf{v} \in \mathsf{M}_{\mathbb{K}}} \max\left(1, |\beta|_{\mathsf{v}}\right)^{d_{\mathsf{v}}/[\mathbb{K}:\mathbb{Q}]} .$$

Given a number field $\mathbb{K}$, we have the fundamental *Liouville inequality* (cf. [61, p. 82]):

$$\log|\beta| \geq -[\mathbb{K}:\mathbb{Q}]\log H(\beta)\,, \qquad \forall \beta \in \mathbb{K}^\star\,. \tag{4.2}$$

4.6. **Arithmetic progressions.** An *infinite arithmetic progression* is a set of the form $a + b\mathbb{N}$ with $a, b \in \mathbb{N}$ and $b \neq 0$. An arithmetic progression of length $n \geq 1$ is a finite set of the form

$$\{a, a+b, a+2b, \cdots, a+(n-1)b\}$$

with $a, b \in \mathbb{N}$ and $b \neq 0$.

5. Admissibility conditions

A well-known feature of Mahler's method is that, independently of the choice of the matrix $A(\boldsymbol{z})$ defining the system (3.1), some unavoidable restrictions on the transformation T and on the point $\boldsymbol{\alpha}$ are required.

Definition 5.1. Let $T \in \mathcal{M}_n(\mathbb{N})$ and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$. The pair $(T, \boldsymbol{\alpha})$ is said to be *admissible* if there exist two real numbers $\rho > 1$ and $c > 0$ such that the following three conditions hold.

(a) The coefficients of the matrix T^k belong to $\mathcal{O}(\rho^k)$.
(b) Set $T^k\boldsymbol{\alpha} =: (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$. Then $\log|\alpha_i^{(k)}| \leq -c\rho^k$, for every integer i, $1 \leq i \leq n$, and all sufficiently large integers k.
(c) If $f(\boldsymbol{z}) \in \mathbb{C}\{\boldsymbol{z}\}$ is nonzero, then there are infinitely many integers k such that $f(T^k\boldsymbol{\alpha}) \neq 0$.

The strength of our results strongly depends on our ability to provide simple and natural conditions that imply Conditions (a), (b), and (c) as they are necessary to apply Mahler's method (see [39]). Though they appear naturally in proofs, it is not that easy, at first glance, to see how to check them. We provide here a simple characterization of matrices and algebraic points satisfying these conditions, gathering results of Kubota [33], Loxton and van der Poorten [35, 36], and mainly Masser [43].

Definition 5.2. Let $T \in \mathcal{M}_n(\mathbb{N})$ with spectral radius $\rho(T)$. We say that T belongs to the class $\mathcal{T}$ if it satisfies the following three conditions.

(i) It is nonsingular.
(ii) None of its eigenvalues are roots of unity.
(iii) It has a *Perron-Frobenius eigenvector*, that is an eigenvector with real positive coefficients associated with the eigenvalue $\rho(T)$.

Remark 5.3. If $T \in \mathcal{T}$, then $\rho(T) > 1$.

Let us recall some basic facts about matrices with nonnegative real coefficients that can be found in [30]. Let T be such a matrix. Then its spectral radius $\rho(T)$ is an eigenvalue of T and T always has an associated eigenvector with nonnegative real coefficients. When such a vector has positive coefficients, it is called a *Perron-Frobenius eigenvector*. A square matrix with nonnegative real coefficients is *irreducible* if there is no permutation of the rows and the columns such that it has a block decomposition of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} .$$

It follows from the Perron-Frobenius theorem [30, Chapter 2, Theorem 2] that any irreducible matrix has a Perron-Frobenius eigenvector. In particular, it satisfies Condition (iii) of Definition 5.2. Any matrix T with nonnegative real coefficients, after a permutation of its rows and columns, may be written under the form

$$\begin{pmatrix} T_1 & & & \\ S_{2,1} & T_2 & & \\ \vdots & \ddots & \ddots & \\ S_{\mu,1} & \cdots & S_{\mu,\mu-1} & T_\mu \end{pmatrix} , \tag{5.1}$$

where $T_1, ..., T_\mu$ are square matrices. We say that a block T_i is *isolated* if $S_{i,j} = 0$ for all $j < i$. There exists such a decomposition in which $T_1, \ldots, T_\mu$ are irreducible matrices and where there exists an integer $\kappa \geq 1$ such that the blocks $T_1, \ldots, T_\kappa$ are isolated, while none of the blocks $T_{\kappa+1}, \ldots, T_\mu$ is isolated. Such a decomposition is called the *normal form* of T. From [30, Chapter 3, Theorem 6], T satisfies Condition (iii) of Definition 5.2 if and only if $\rho(T_1) = \cdots = \rho(T_\kappa) = \rho(T)$ and $\rho(T_i) < \rho(T)$ for every i such that $\kappa + 1 \leq i \leq \mu$.

Remark 5.4. This discussion shows that there is no difficulty in checking whether or not a given matrix T belongs to the class $\mathcal{T}$. If $T_1, \ldots, T_r \in \mathcal{T}$ have the same spectral radius, then $T_1 \oplus \cdots \oplus T_r \in \mathcal{T}$. More generally, if $T_1, \ldots, T_r \in \mathcal{T}$ have pairwise multiplicatively dependent spectral radii, then there exist positive integers $a_1, \ldots, a_r$, such that $T_1^{a_1} \oplus \cdots \oplus T_r^{a_r} \in \mathcal{T}$.

Example 5.5. The matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

belong to $\mathcal{T}$, while

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} ,$$

do not belong to $\mathcal{T}$ for they respectively do not satisfy Condition (i), (ii), or (iii) in Definition 5.2.

Let us note the following results for future use.

Lemma 5.6. *Let $T \in \mathcal{T}$. Then, there exists a real number $\kappa > 0$ such that for every row vector with nonnegative real coefficients $\boldsymbol{\lambda}$ and every integer $k \geq 0$,*

$$|\boldsymbol{\lambda} T^k| \geq \kappa \rho(T)^k |\boldsymbol{\lambda}| .$$

Proof. Let $\boldsymbol{\mu}$ be a Perron-Frobenius eigenvector of T with $\|\boldsymbol{\mu}\| \leq 1$ and let $\kappa > 0$ denote the smallest coordinate of $\boldsymbol{\mu}$. Following the notation in Section 4.1, we have $T(\boldsymbol{\mu}) = \rho(T)\boldsymbol{\mu}$. Then

$$|\boldsymbol{\lambda} T^k| \geq |\boldsymbol{\lambda} T^k(\boldsymbol{\mu})| = \rho(T)^k |\boldsymbol{\lambda}\boldsymbol{\mu}| \geq \kappa \rho(T)^k |\boldsymbol{\lambda}| ,$$

as wanted. □

Lemma 5.7. *Let $T \in \mathcal{M}_n(\mathbb{N})$ be a non-singular matrix, and let h be a positive integer. If T^h admits a Perron-Frobenius eigenvector, then T also has a Perron-Frobenius eigenvector.*

Proof. Let (5.1) be the normal form of T. Then the matrix T^h admits a decomposition of the form

$$\begin{pmatrix} T_1^h & & & & & \\ & \ddots & & & & \\ & & T_\kappa^h & & & \\ S_{\kappa+1,1}^{(h)} & \cdots & S_{\kappa+1,\kappa}^{(h)} & T_\kappa^h & & \\ \vdots & & & \ddots & \ddots & \\ S_{\mu,1}^{(h)} & \cdots & \cdots & \cdots & S_{\mu,\mu-1}^{(h)} & T_\mu^h \end{pmatrix}, \tag{5.2}$$

where the unspecified blocks are zero. Since this matrix has non-negative entries and the matrices T_i are non-singular, it follows that none of the blocks $T_{\kappa+1}^h, \ldots, T_\mu^h$ is isolated. It follows from [30, Chapter 3, Theorem 9] that, up to a permutation, we have

$$T_i^h = \operatorname{diag}(T_{i,1}^{(h)}, \ldots, T_{i,s_i}^{(h)}), \quad \forall i,\, 1 \leq i \leq \mu\,,$$

where each $T_{i,j}^{(h)}$ is irreducible, and $\rho(T_{i,1}^{(h)}) = \cdots = \rho(T_{i,s_i}^{(h)}) = \rho(T_i^h)$. Thus, the matrices $T_{i,j}^{(h)}$, $1 \leq i \leq \mu$, $1 \leq j \leq s_i$, are the diagonal blocks in the normal form of T^h. Suppose that T^h has a Perron-Frobenius eigenvector. Since the blocks $T_{i,j}^{(h)}$, $1 \leq i \leq \kappa$, are isolated in the normal form of T^h, we obtain

$$\rho(T^h) = \rho(T_{i,j}^{(h)}) = \rho(T_i^h)\,.$$

This implies that $\rho(T_i) = \rho(T) = \rho$ for all $i \leq \kappa$. Furthermore, for each $i \geq \kappa + 1$, since T_i^h is not isolated in (5.2), there exists an integer j such that the block $T_{i,j}^{(h)}$ is not isolated in the normal form of T^h. As T^h has a Perron-Frobenius eigenvector, we have $\rho(T_i^h) \leq \rho(T_{i,j}^{(h)}) < \rho(T^h) = \rho^h$. Thus, $\rho(T_i) < \rho$. According to [30, Chapter 3, Theorem 6], we deduce that T has a Perron-Frobenius eigenvector. □

Given a one-variable Mahler system associated with a matrix $A(z)$, we can consider the same system twice but with different variables. That is, the system associated with the matrix

$$\begin{pmatrix} A(z_1) & 0 \\ 0 & A(z_2) \end{pmatrix}.$$

This shows that some kind of minimal independence between the coordinates of the point $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is required in order to apply Mahler's method. Typically, we cannot consider a point of the form (α, α) in that case.

Definition 5.8. A point $\boldsymbol{\alpha} \in (\mathbb{C}^\star)^n$ is said to be T-*independent* if there is no nonzero n-tuple of integers $\boldsymbol{\mu}$ for which $(T^k\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 1$ for all k in an infinite arithmetic progression.

Remark 5.9. If $\alpha_1,\ldots,\alpha_n \in \mathbb{C}^\star$ are multiplicatively independent complex numbers, then $(\alpha_1,\ldots,\alpha_n)$ is T-independent for all $T \in \mathcal{T}$. According to Definition 5.8, Condition (ii) of Theorem 3.6 is equivalent to the fact that the point $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r)$ is T-independent with respect to the direct sum $T := T_1 \oplus \cdots \oplus T_r$.

With these definitions, we have the following characterization of admissibility, which makes our main results very convenient to apply.

Theorem 5.10. *Let $T \in \mathcal{M}_n(\mathbb{N})$ and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$. Then the following properties are equivalent.*

(i) *The pair $(T,\boldsymbol{\alpha})$ is admissible.*

(ii) *$T \in \mathcal{T}$, $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = 0$, and $\boldsymbol{\alpha}$ is T-independent.*

Proof of Theorem 5.10. We first prove that (ii) implies (i). Let us assume that T belongs to $\mathcal{T}$, $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is T-independent, and that $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = 0$. Then there exists k_0 such that $\|T^{k_0}\boldsymbol{\alpha}\| < 1$. We observe that if the pair $(T, T^{k_0}\boldsymbol{\alpha})$ is admissible, then so is the pair $(T,\boldsymbol{\alpha})$. Thus, we can assume without any loss of generality that $\|\boldsymbol{\alpha}\| < 1$. We set

$$\boldsymbol{x} := {}^{\mathrm{t}}(-\log|\alpha_1|,\ldots,-\log|\alpha_n|)\,, \tag{5.3}$$

so that $\boldsymbol{x}$ is a column vector with positive coordinates. By assumption, T has a positive eigenvector associated with the eigenvalue $\rho(T)$. Let us choose such an eigenvector $\boldsymbol{\mu}$ whose coordinates are all smaller than those of $\boldsymbol{x}$. According to the notation of Section 4.1, we use $T\boldsymbol{\alpha}$ for the nonlinear action of T and $T(\boldsymbol{x})$ for the usual linear action of T. Then we have

$$-\log\|T^k\boldsymbol{\alpha}\| = \|T^k(\boldsymbol{x})\| = \|T^k(\boldsymbol{\mu}) + T^k(\boldsymbol{x}-\boldsymbol{\mu})\| > \|T^k(\boldsymbol{\mu})\| = \rho(T)^k\|\boldsymbol{\mu}\|\,,$$

for all $k \in \mathbb{N}$, because $T^k(\boldsymbol{x}-\boldsymbol{\mu})$ has positive coordinates. Condition (b) is thus satisfied with $\rho = \rho(T)$. By [35, Lemma 3], there exists a positive integer h such that $\|T^{hk}\| = \mathcal{O}(\rho(T)^{hk})$. Then,

$$\|T^k\| \leq n\|T^{h-1}\| \times \|T^{h\lfloor k/h\rfloor}\| = \mathcal{O}(\rho(T)^{h\lfloor k/h\rfloor}) = \mathcal{O}(\rho(T)^k)\,.$$

Thus, Condition (a) is satisfied. Finally, Masser vanishing theorem [43] implies that Condition (c) holds since $\boldsymbol{\alpha}$ is T-independent. Hence, the pair $(T,\boldsymbol{\alpha})$ is admissible.

Now, we prove that (i) implies (ii). Let $(T,\boldsymbol{\alpha})$ be an admissible pair. First, we note that Condition (b) implies that $\lim_{k\to\infty} T^k\boldsymbol{\alpha} = 0$. Secondly, we note that Condition (c) implies that $\boldsymbol{\alpha}$ is T-independent. Indeed, otherwise there would exist two distinct tuples of nonnegative integers $(s_1,\ldots,s_n)$ and $(t_1,\ldots,t_n)$ such that $P(T^k\boldsymbol{\alpha}) = 0$ for infinitely many k, where $P(\boldsymbol{z}) := z_1^{s_1}\cdots z_n^{s_n}$ - $z_1^{t_1}\cdots z_n^{t_n}$, providing a contradiction with Condition (c). It thus remains to prove that $T \in \mathcal{T}$. Since it is proved in [33] that Condition (c) implies that the matrix T is nonsingular and that none of its eigenvalues is a root of unity, it only remains to prove that T has a Perron-Frobenius eigenvector.

Replacing $\boldsymbol{\alpha}$ by $T^k\boldsymbol{\alpha}$ for some k if necessary, we can assume without any loss of generality that $\|\boldsymbol{\alpha}\| < 1$. As noticed in Remark 5.3, $\rho(T) > 1$. Let ρ

be as in Definition 5.1. By Gelfand's formula, $\rho(T) = \lim_{k\to\infty} \|T^k\|^{1/k}$ and then Condition (a) implies that $\rho(T) \le \rho$. On the other hand, setting

$$c_0 := \min(-\log|\alpha_i| : 1 \le i \le n) > 0\,,$$

we have

$$\log|\alpha_i^{(k)}| \ge -nc_0\|T^k\|\,, \quad \forall i\,,\ 1 \le i \le n\,,\ \forall k \ge 0\,.$$

We thus infer from Condition (b) and Gelfand's formula that $\rho \le \rho(T)$, so that $\rho = \rho(T)$. Since the coefficients of T are nonnegative integers, for every eigenvalue ρ' with $|\rho'| = \rho$, there is a root of unity μ such that $\rho' = \mu\rho$ (cf. [30, Theorem 2, p. 65 & Chapter III §4]). Replacing T by a suitable power of T if necessary, which we may do without loss of generality by Lemma 5.7, we can assume that ρ is larger than all other eigenvalues of T. Let E_ρ denote the eigenspace associated with ρ. Condition (a) implies that the characteristic space associated with ρ is equal to E_ρ, since otherwise the sequence T^k/ρ^k would not be bounded. Hence E_ρ has a T-invariant vector space complement, say E_ρ^c. From Condition (b), we infer the existence of a real number $\gamma > 0$ such that every coordinate of $T^k(\boldsymbol{x})$ is larger than $\gamma\rho^k$, where $\boldsymbol{x}$ is defined in (5.3). We have a decomposition

$$\boldsymbol{x} = \boldsymbol{e} + \boldsymbol{e}^c, \text{ where } \boldsymbol{e} \in E_\rho \text{ and } \boldsymbol{e}^c \in E_\rho^c\,.$$

Suppose that there is no vector in E_ρ with positive coordinates. Then, for some j, the jth coordinate of $\boldsymbol{e}$ is nonpositive. Since $T^k(\boldsymbol{x}) = \rho^k\boldsymbol{e} + T^k(\boldsymbol{e}^c)$, we deduce that the jth coordinate of $T^k(\boldsymbol{e}^c)$ is larger than $\gamma\rho^k$. Since all eigenvalues of T on E_ρ^c are smaller than ρ, we obtain a contradiction. This shows that T has a Perron-Frobenius eigenvector and completes the proof. □

6. A new vanishing theorem

As already mentioned, it is of great importance to find natural conditions that ensure nonvanishing properties similar to Condition (c) in Definition 5.1. Of course, our goal is to obtain a *vanishing theorem* that can be applied to transformation matrices and points which are as general as possible. Our contribution to this problem is Theorem 6.4.

6.1. **Piecewise syndetic, full, and negligible sets.** In the framework of Mahler's method, several vanishing theorems have been formulated by saying that a nonzero multivariate power series cannot vanish at all points in some well-structured large sets. The latter are obtained by iteration of the transformation matrix and usually involve arithmetic progressions. In order to prove our main theorems, we need to replace these *well-structured sets* by sets which remain large but offer much more flexibility. We use the notion of piecewise syndetic set, which is classical in Ramsey theory, especially in its ergodic counterpart. As we just said, it can be thought of as a notion of largeness for subsets of $\mathbb{N}$. Furthermore, Brown's lemma (see (ii) in Lemma 6.2) shows that such sets are partition regular, and thus much more robust in terms of partitions than arithmetic progressions.

Definition 6.1. A set $\mathcal{Z} \subset \mathbb{N}$ is said to be *piecewise syndetic* if there exists a natural number $B \geq 1$ such that for any given integer $C \geq 2$ there exist $l_1 < \cdots < l_C$ in $\mathcal{Z}$ such that

$$l_{i+1} - l_i \leq B, \quad 1 \leq i < C\,.$$

In this case, we say that B is a *bound* for $\mathcal{Z}$. A set $\mathcal{Z} \subset \mathbb{N}$ is said to be *negligible* if it is not piecewise syndetic, while it is said to be *full* if $\mathbb{N} \setminus \mathcal{Z}$ is negligible.

Let us recall that a subset of $\mathbb{N}$ is said to be *syndetic*, or sometimes *relatively dense*, if it has bounded gaps. A subset of $\mathbb{N}$ is said to be *thick* if it contains arbitrarily long intervals. Thus piecewise syndetic sets are those that can be obtained as the intersection of a syndetic set and a thick set. In the rest of this section, as well as all along Section 8, we will use heavily the following results.

Lemma 6.2. *Let $\mathcal{Z} \subset \mathbb{N}$ be a piecewise syndetic set with bound B. Then the following properties hold.*

(i) *If $\mathcal{Z} \subset \mathcal{Z}' \subset \mathbb{N}$, then $\mathcal{Z}'$ is also piecewise syndetic.*
(ii) *If $\mathcal{Z} \subset \cup_{i=1}^{s} \mathcal{Z}_i$, then at least one of the sets $\mathcal{Z}_i$ is piecewise syndetic.*
(iii) *Let $m_0 \in \mathbb{N}$. The set*

$$\mathcal{Z}_0 := \{l \in \mathcal{Z} : \exists m \in [m_0, m_0 + B - 1] \text{ such that } l + m \in \mathcal{Z}\}$$

is piecewise syndetic.
(iv) *The set $\mathcal{Z}$ contains arbitrarily long arithmetic progressions.*
(v) *Let $\pi : \mathcal{Z} \to \mathbb{N}$ be such that $|\pi(l) - l|$ is bounded. The set*

$$\pi(\mathcal{Z}) = \{\pi(l) \,:\, l \in \mathcal{Z}\}$$

is piecewise syndetic.

Proof. Property (i) immediately follows from the definition, while Properties (ii) and (iv) correspond to classical results respectively known as Brown's lemma (see [20]) and Szemerédi's theorem [60].

Let us prove (iii). Let m_0 and $C \geq 2$ be two natural numbers and let a be the smallest integer such that $aB > m_0$. Since $\mathcal{Z}$ is piecewise syndetic, there exists a sequence $l_1 < l_2 < \cdots < l_{C+aB}$ of integers in $\mathcal{Z}$ such that $l_{i+1} - l_i < B$. Let $i \in \{1, \ldots, C\}$. Then, $l_i + m_0 < l_i + aB \leq l_{i+aB}$. There thus exists an integer $j \leq aB$ such that $l_i + m_0 \leq l_{i+j} \leq l_i + m_0 + B - 1$. Hence $l_i \in \mathcal{Z}_0$. Thus, $l_1, \ldots, l_C$ all belong to the set $\mathcal{Z}_0$, which proves that this set is piecewise syndetic.

Let us prove (v). Let $C \geq 2$ be a natural number and let a be an integer such that $|\pi(l) - l| < a$ for every $l \in \mathcal{Z}$. If $l, l' \in \mathcal{Z}$ and $l' \geq l + 2a$, we have $\pi(l) < \pi(l')$. Since $\mathcal{Z}$ is piecewise syndetic, there exists a sequence $l_0 < l_1 < \cdots < l_{2aC} \in \mathcal{Z}$ with $l_{i+1} - l_i < B$. Let $k_i := \pi(l_{2ai})$, $0 \leq i \leq C$. Then, $k_i \in \pi(\mathcal{Z})$. Since for every i, $l_{2a(i+1)} \geq l_{2ai} + 2a$, we have $k_0 < k_1 < \cdots < k_C$. On the other hand, for every i, $0 \leq i \leq C$, we have

$$\begin{aligned} k_{i+1} - k_i =& \pi(l_{2a(i+1)}) - \pi(l_{2ai}) \\ \leq& |\pi(l_{2a(i+1)}) - l_{2a(i+1)}| + |l_{2a(i+1)} - l_{2ai}| + |l_{2ai} - \pi(l_{2ai})| \\ \leq& 2a(B+1) \end{aligned}$$

Thus, $\pi(\mathcal{Z})$ is a piecewise syndetic set with bound $2a(B+1)$. ∎

Part of Lemma 6.2 can be naturally rephrased as follows.

Lemma 6.3. *The following properties hold.*

(i) *A subset of a negligible set is negligible.*
(ii) *A finite union of negligible sets is negligible.*
(iii) *A finite intersection of full sets is full.*
(iv) *If $\mathcal{Z}_1$ is full and $\mathcal{Z}_2$ is negligible, then $\mathcal{Z}_1 \setminus \mathcal{Z}_2$ is full.*

Proof. Property (i) follows directly from Property (i) of Lemma 6.2. Property (ii) follows directly from Property (ii) of Lemma 6.2. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_r$ be full sets. For a set $\mathcal{Z} \subset \mathbb{N}$, we let $\mathcal{Z}^c$ denote the set $\mathbb{N} \setminus \mathcal{Z}$. By (ii), we obtain that $(\cap \mathcal{Z}_i)^c = \cup(\mathcal{Z}_i^c)$ is negligible, which proves (iii). Let us prove (iv). By assumption, $\mathcal{Z}_1^c$ is negligible. By (ii), the set $\mathcal{Z}_2 \cup \mathcal{Z}_1^c$ is also negligible. Since $\mathcal{Z}_2 \cup \mathcal{Z}_1^c = (\mathcal{Z}_1 \setminus \mathcal{Z}_2)^c$, we obtain that $\mathcal{Z}_1 \setminus \mathcal{Z}_2$ is full, as wanted. □

6.2. **The vanishing theorem.** We are now ready to state and prove our vanishing theorem.

Theorem 6.4. *Let $T_1, \ldots, T_r$ be matrices in the class $\mathcal{T}$ whose spectral radii $\rho(T_1), \ldots, \rho(T_r)$ are pairwise multiplicatively independent. Let n_i denote the size of the matrix T_i and set $N := \sum_{i=1}^r n_i$. Set*

$$\Theta := \left(\frac{1}{\log \rho(T_1)}, \ldots, \frac{1}{\log \rho(T_r)} \right). \tag{6.1}$$

Let $(\boldsymbol{k}_l = (k_{1,l}, \ldots, k_{r,l}))_{l \in \mathbb{N}}$ denote a sequence of r-tuple of positive integers such that

$$\boldsymbol{k}_l = l\Theta + \mathcal{O}(1). \tag{6.2}$$

Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r) \in (\overline{\mathbb{Q}}^\star)^N$ be such that the pair $(T_i, \boldsymbol{\alpha}_i)$ is admissible for every i, $1 \le i \le r$, and let $g(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ be nonzero. Then the set

$$\left\{ l \in \mathbb{N} : g(T_1^{k_{1,l}} \boldsymbol{\alpha}_1, \ldots, T_r^{k_{r,l}} \boldsymbol{\alpha}_r) = 0 \right\}$$

is negligible.

Applying Mahler's method to several Mahler systems requires some uniform speed of convergence to the origin for the orbits of each algebraic point $\boldsymbol{\alpha}_i$ under the matrix transformations T_i. As noticed by van der Poorten [55], one way to overcome this difficulty is to iterate each transformation T_i k_i-times, and to choose the iteration vector $\boldsymbol{k} = (k_1, \ldots, k_r)$ so that asymptotically the matrices $T_i^{k_i}$ have essentially the same radius of convergence. This explains why the assumption (6.2) is natural in this framework. In the rest of this section, we set

$$T_{\boldsymbol{k}} := T_1^{k_1} \oplus \cdots \oplus T_r^{k_r} \quad \text{so that} \quad T_{\boldsymbol{k}} \boldsymbol{\alpha} = (T_1^{k_1} \boldsymbol{\alpha}_1, \ldots, T_r^{k_r} \boldsymbol{\alpha}_r).$$

Lemma 6.5. *We continue with the notation and assumptions of Theorem 6.4. There exists a real number $c > 0$ such that*

$$\|T_{\boldsymbol{k}_l}\| = \mathcal{O}(e^l) \quad \text{and} \quad \log \|T_{\boldsymbol{k}_l} \boldsymbol{\alpha}\| \le -ce^l, \quad \forall l \gg 1.$$

Proof. From (6.2), there exists a positive real number B, such that, for every $l \in \mathbb{N}$ and i, $1 \le i \le r$, we have $k_{i,l} = l / \log(\rho(T_i)) + \varepsilon(i, l)$, for some real number $\varepsilon(i,l)$ with $|\varepsilon(i,l)| \le B$. By Conditions (a) and (b) in Definition 5.1, we have, on the one hand, that

$$\left\|T_i^{k_{i,l}}\right\| = \mathcal{O}\left(\rho(T_i)^{k_{i,l}}\right) = \mathcal{O}\left(\rho(T_i)^{l/\log(\rho(T_i))+\varepsilon(i,l)}\right) = \mathcal{O}\left(e^l\right),$$

while, on the other hand,

$$\log\left\|T_i^{k_{i,l}}\boldsymbol{\alpha}_i\right\| \le -c_i\rho(T_i)^{k_{i,l}} \le -c_i\rho(T_i)^{-B}e^l,$$

for all l large enough, where c_i are positive real numbers. Setting $c := \min\{c_1\rho(T_1)^{-B},\dots,c_r\rho(T_r)^{-B}\}$, we obtain the desired estimate. □

Before proving Theorem 6.4, we need the two following auxiliary results. The proof of Theorem 6.4 is based on a vanishing theorem due to Corvaja and Zannier [24, Theorem 3]. The latter states that if the set of zeros of a multivariate analytic function with algebraic coefficients contains an infinite sequence of $\mathcal{S}$-unit points whose height does not grow too fast, then these points all belong to a finite number of translates of tori. The goal of the following two lemmas is to show that most points of the form $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$, $l \in \mathbb{N}$, avoid these tori.

Lemma 6.6. *We continue with the assumptions of Theorem 6.4. Let $\boldsymbol{\mu}$ be a nonzero N-tuple of integers. Then the set*

$$\mathcal{Z} := \{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 1\}$$

is negligible.

Proof. We argue by contradiction, assuming that $\mathcal{Z}$ is piecewise syndetic. For every pair of nonnegative integers (l,m), with $m > 0$, we define the r-tuple of integers $\boldsymbol{e} = \boldsymbol{e}(l,m)$ by

$$\boldsymbol{e} = \boldsymbol{k}_{l+m} - \boldsymbol{k}_l \tag{6.3}$$

and we set $\mathcal{E} := \{\boldsymbol{e}(l,m) :\ l,\, m \in \mathbb{N}\}$. Since by (6.2) we have $\boldsymbol{k}_l = l\Theta + \mathcal{O}(1)$, we obtain that

$$\boldsymbol{e}(l,m) = m\Theta + \mathcal{O}(1), \tag{6.4}$$

which shows that $\mathcal{E}$ is infinite. However, given any positive integer m_0, the set $\{\boldsymbol{e}(l,m_0) :\ l \in \mathbb{N}\}$ is finite.

Let us remark that given any pair (β_1,β_2) of nonzero complex numbers that are not roots of unity, and any pair of natural numbers (i,j), $1 \le i < j \le r$, the set

$$\mathcal{E}_1 := \left\{\boldsymbol{e} = (e_1,\dots,e_r) \in \mathcal{E} : \beta_1^{e_i} = \beta_2^{e_j}\right\}$$

is finite. Indeed, the set of integers u such that there exists an integer v for which $\beta_1^u = \beta_2^v$ is an ideal of $\mathbb{Z}$. Let $u_0 \ge 0$ be a generator of this ideal and let $v_0 \in \mathbb{Z}$ be such that $\beta_1^{u_0} = \beta_2^{v_0}$. Write $\mathcal{E}_1 = \mathcal{E}_2 \cup \mathcal{E}_3$ where

$$\mathcal{E}_2 := \{\boldsymbol{e} \in \mathcal{E}_1 : e_i = 0\} \text{ and } \mathcal{E}_3 := \{\boldsymbol{e} \in \mathcal{E}_1 : e_i \neq 0\}.$$

It follows from (6.4) that the set $\mathcal{E}_2$ is finite. Thus, we only have to prove that $\mathcal{E}_3$ is finite. If $u_0 = 0$, then $e_i = 0$ for every $\boldsymbol{e} \in \mathcal{E}_1$. Thus $\mathcal{E}_3 = \emptyset$ is a finite set. Suppose that $u_0 > 0$. For every $(e_1,\dots,e_r) \in \mathcal{E}_3$, there exists an integer $a \neq 0$ such that $e_i = au_0$. We obtain that

$$\beta_2^{e_j} = \beta_1^{e_i} = \beta_1^{au_0} = \beta_2^{av_0}.$$

Since β_2 is nonzero and is not a root of unity, we have $e_j = av_0$. Hence $e_j/e_i = v_0/u_0 \in \mathbb{Q}$. Since $e_i = k_{i,l+m} - k_{i,l}$ and $e_j = k_{j,l+m} - k_{j,l}$, we get that

$$\frac{k_{i,l+m} - k_{i,l}}{k_{j,l+m} - k_{j,l}} = \frac{v_0}{u_0} \in \mathbb{Q}\,.$$

Now let us assume by contradiction that $\mathcal{E}_3$ is infinite. Then, there exist arbitrarily large integers m with this property. Letting m tends to infinity, we deduce from (6.4) that the ratio $\log\rho(T_i)/\log\rho(T_j)$ is rational. This provides a contradiction since by assumption $\rho(T_i)$ and $\rho(T_j)$ are multiplicatively independent. Hence $\mathcal{E}_3$ is finite and so is $\mathcal{E}_1$.

Let us recall that, by assumption, none of the eigenvalues of the matrices T_i is equal to zero or to a root of unity. The previous reasoning shows that there exists a positive integer m_0 such that, for every $m \geq m_0$, every $l \in \mathbb{N}$, every eigenvalue λ_i of T_i, and every eigenvalue λ_j of T_j, $i \neq j$, we have

$$\lambda_i^{e_i} \neq \lambda_j^{e_j}\,, \tag{6.5}$$

where $\boldsymbol{e} = \boldsymbol{e}(l,m) = (e_1, \ldots, e_r)$. For such a vector $\boldsymbol{e}$, set

$$T_{\boldsymbol{e}} := T_1^{e_1} \oplus \cdots \oplus T_r^{e_r}\,.$$

Given a vector space $V \subset \mathbb{C}^N$, we have

$$V \subset \bigoplus_{i=1}^{r} \iota_i \circ \pi_i(V)\,,$$

where we let $\pi_i : \mathbb{C}^N = \mathbb{C}^{n_1+\cdots+n_r} \to \mathbb{C}^{n_i}$ denote the projection defined by $\pi_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r) = \boldsymbol{x}_i$ and $\iota_i : \mathbb{C}^{n_i} \to \mathbb{C}^N$ be the canonical injection with respect to the decomposition $\mathbb{C}^N = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}$. By (6.5), if V is invariant under $T_{\boldsymbol{e}}$, then

$$V = \bigoplus_{i=1}^{r} \iota_i \circ \pi_i(V)\,. \tag{6.6}$$

We are now ready to proceed with the proof of the lemma. Let $\mathbb{K}$ be a number field containing the coordinates of $\boldsymbol{\alpha}$ and $\mathsf{M}_{\mathbb{K}}$ be the set of places of $\mathbb{K}$. For every $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, we set

$$\boldsymbol{x}_{\mathsf{v}} := {}^{\mathrm{t}}(\log|\alpha_{1,1}|_{\mathsf{v}}, \ldots, \log|\alpha_{1,n_1}|_{\mathsf{v}}, \log|\alpha_{2,1}|_{\mathsf{v}}, \ldots, \log|\alpha_{r,n_r}|_{\mathsf{v}})\,, \tag{6.7}$$

By assumption, we have

$$\langle \boldsymbol{\mu}\,, T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})\rangle = 0$$

for all $l \in \mathcal{Z}$ and all $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, where we let $\langle\,,\rangle$ denote the usual scalar product. We recall that, according to the notation of Section 4.1, $T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})$ (with parentheses) stands for the usual product between the matrix $T_{\boldsymbol{k}_l}$ and the column vector $\boldsymbol{x}_{\mathsf{v}}$. Let U denote the orthogonal complement to the vector $\boldsymbol{\mu}$ in $\mathbb{C}^N$. This is a proper subspace of $\mathbb{C}^N$ defined over $\mathbb{Q}$, which contains all vectors $T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})$, $l \in \mathcal{Z}$, $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$. Given $\mathcal{Z}' \subset \mathcal{Z}$, we let $U(\mathcal{Z}')$ denote the smallest vector subspace of $\mathbb{C}^N$ defined over $\mathbb{Q}$ and containing all $T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})$, $l \in \mathcal{Z}'$, $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$. It follows that $U(\mathcal{Z}) \subset U$. Furthermore, if $\mathcal{Z}'' \subset \mathcal{Z}'$, then $U(\mathcal{Z}'') \subset U(\mathcal{Z}')$. The subspace $U(\mathcal{Z})$ having finite dimension and $\mathcal{Z}$ being piecewise syndetic, there exists a subset $\mathcal{Z}_1 \subset \mathcal{Z}$ that is piecewise syndetic, and such that for all piecewise syndetic sets $\mathcal{Z}' \subset \mathcal{Z}_1$, one has $U(\mathcal{Z}') = U(\mathcal{Z}_1)$. Let B denote a bound for $\mathcal{Z}_1$ and set

$$\mathcal{E}_0 := \{\boldsymbol{e}(l,m) : l \in \mathcal{Z}_1, l+m \in \mathcal{Z}_1, m \in [m_0, m_0+B-1]\} ,$$

where m_0 is defined as in the first part of the proof (just before (6.5)). This is a finite set. Let

$$\mathcal{Z}_2 := \{l \in \mathcal{Z}_1 : \exists m \in [m_0, m_0+B-1] \text{ such that } l+m \in \mathcal{Z}_1\} .$$

By Property (iii) of Lemma 6.2, the set $\mathcal{Z}_2$ is piecewise syndetic. Now, given $\boldsymbol{e} \in \mathcal{E}_0$, we set

$$\mathcal{Z}_{\boldsymbol{e}} := \{l \in \mathcal{Z}_2 : \exists m \in [m_0, m_0+B-1] \text{ such that } l+m \in \mathcal{Z}_1 \text{ and } \boldsymbol{e}(l,m) = \boldsymbol{e}\}$$

so that $\mathcal{Z}_2 = \cup_{\boldsymbol{e}\in\mathcal{E}_0} \mathcal{Z}_{\boldsymbol{e}}$. Since $\mathcal{Z}_2$ is piecewise syndetic, Property (ii) of Lemma 6.2 ensures the existence of $\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathcal{E}_0$ such that $\mathcal{Z}_{\boldsymbol{e}}$ is piecewise syndetic. Furthermore, $\mathcal{Z}_{\boldsymbol{e}} \subset \mathcal{Z}_1$. Thus we obtain that

$$U(\mathcal{Z}_{\boldsymbol{e}}) = U(\mathcal{Z}_1) .$$

We claim that the vector space $U(\mathcal{Z}_1)$ is invariant under $T_{\boldsymbol{e}}$. Indeed, since $U(\mathcal{Z}_1) = U(\mathcal{Z}_{\boldsymbol{e}})$, it is enough to prove that for any $l \in \mathcal{Z}_{\boldsymbol{e}}$ and any $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, $T_{\boldsymbol{e}}(T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})) \in U(\mathcal{Z}_1)$. Let $l \in \mathcal{Z}_{\boldsymbol{e}}$. Then, there exists $m \in [m_0, m_0+B-1]$ such that $l+m \in \mathcal{Z}_1$ and $\boldsymbol{k}_{l+m} - \boldsymbol{k}_l = \boldsymbol{e}$. Thus $T_{\boldsymbol{e}}(T_{\boldsymbol{k}_l}(\boldsymbol{x}_{\mathsf{v}})) = T_{\boldsymbol{k}_{l+m}}(\boldsymbol{x}_{\mathsf{v}}) \in U(\mathcal{Z}_1)$ for all $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, as wanted. Hence, $U(\mathcal{Z}_1)$ is invariant under $T_{\boldsymbol{e}}$. By (6.6), there is a decomposition of the form

$$U(\mathcal{Z}_1) = \bigoplus_{i=1}^{r} \iota_i(U_i) ,$$

where, for every i, $U_i = \pi_i(U(\mathcal{Z}_1)) \subset \mathbb{C}^{n_i}$. Since $U(\mathcal{Z}_1)$ is a proper subspace of $\mathbb{C}^N$, there exists i, $1 \leq i \leq r$, such that U_i is a proper subspace of $\mathbb{C}^{n_i}$. This vector space being defined over $\mathbb{Q}$, it has a nonzero vector $\boldsymbol{\nu}_0 \in \mathbb{Z}^{n_i}$ in its orthogonal complement. For all $l \in \mathcal{Z}_1$ and $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, the vectors $T_i^{k_{i,l}}(\boldsymbol{x}_{i,\mathsf{v}})$ belong to U_i and it follows that

$$\langle \boldsymbol{\nu}_0 \, , \, T_i^{k_{i,l}}(\boldsymbol{x}_{i,\mathsf{v}}) \rangle = 0 \, , \tag{6.8}$$

where $\boldsymbol{x}_{i,\mathsf{v}} := \pi_i(\boldsymbol{x}_{\mathsf{v}})$. The set $\mathcal{Z}_1$ being piecewise syndetic, we infer from (6.2) that the set $\mathcal{Z}_3 := \{k_{i,l} : l \in \mathcal{Z}_1\}$ is also piecewise syndetic. By Property (iv) of Lemma 6.2, it contains arbitrarily long arithmetic progressions. Let us consider an arithmetic progression of length n_i in $\mathcal{Z}_3$, say

$$a, a+b, a+2b, \ldots, a+(n_i-1)b ,$$

where $a, b \in \mathbb{N}$, $b \neq 0$. Let us also consider the sequences of vector spaces

$$V_{0,\mathsf{v}} \subset \cdots \subset V_{n_i-1,\mathsf{v}} \subset \boldsymbol{\nu}_0^{\perp}$$

defined by

$$V_{j,\mathsf{v}} := \mathrm{Vect}_{\mathbb{Q}} \left\{ T_i^{a}(\boldsymbol{x}_{i,\mathsf{v}}), \ldots, T_i^{a+jb}(\boldsymbol{x}_{i,\mathsf{v}}) \right\} .$$

Since $\dim V_{n_i-1,\mathsf{v}} < n_i$ for every $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, there exists j_{v}, $0 \leq j_{\mathsf{v}} < n_i - 1$, such that $V_{j_{\mathsf{v}},\mathsf{v}} = V_{j_{\mathsf{v}}+1,\mathsf{v}}$. The vector space $V_{j_{\mathsf{v}},\mathsf{v}}$ is then invariant under T_i^b

and we get that $\langle \boldsymbol{\nu}_0\,, T_i^{a+kb}(\boldsymbol{x}_{i,\mathsf{v}})\rangle = 0$ for all $k \in \mathbb{N}$. By (6.7), this equality can be rephrased as

$$\left|\left(T_i^{a+kb}\boldsymbol{\alpha}_i\right)^{\boldsymbol{\nu}_0}\right|_{\mathsf{v}} = 1\,, \text{ for all } k \in \mathbb{N}\,.$$

Since this holds for all $\mathsf{v} \in \mathsf{M}_{\mathbb{K}}$, we deduce that $\left(T_i^{a+kb}\boldsymbol{\alpha}_i\right)^{\boldsymbol{\nu}_0}$ is a root of unity for all $k \in \mathbb{N}$. Since all these numbers belong to the number field $\mathbb{K}$, there exists a positive integer d such that they are all dth roots of unity. Consequently, we have

$$\left(T_i^{a+kb}\boldsymbol{\alpha}_i\right)^{d\boldsymbol{\nu}_0} = 1\,, \text{ for all } k \in \mathbb{N}\,.$$

Hence $\boldsymbol{\alpha}_i$ is not T_i-independent. By Theorem 5.10, this provides a contradiction with the assumption that the pair $(T_i, \boldsymbol{\alpha}_i)$ is admissible. □

Lemma 6.7. *We continue with the notation and assumptions of Theorem 6.4. Let $\gamma \in \overline{\mathbb{Q}}^\star$ and $\boldsymbol{\mu}$ be a nonzero N-tuple of integers. Then the set*

$$\mathcal{Z} := \{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}} = \gamma\}$$

is negligible.

Proof. We argue by contradiction, assuming that $\mathcal{Z}$ is piecewise syndetic. Recall that, given two integers l, m, we set $\boldsymbol{e}(l,m) = \boldsymbol{k}_{l+m} - \boldsymbol{k}_l$. It follows from (6.2) that there exists a positive integer m_0 such that $\boldsymbol{e}(l,m)$ has positive coordinates for every $m \geq m_0$ and all $l \in \mathbb{N}$. Let B be a bound for $\mathcal{Z}$ and set

$$\mathcal{E} := \{\boldsymbol{e}(l,m) : l \in \mathcal{Z},\, l+m \in \mathcal{Z},\, m \in [m_0, m_0+B-1]\}\,.$$

This is a finite set. For every $\boldsymbol{e} \in \mathcal{E}$, set

$$\mathcal{Z}_{\boldsymbol{e}} := \left\{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}-\boldsymbol{\mu}T_{\boldsymbol{e}}} = 1\right\}$$

and

$$\mathcal{Z}' := \{l \in \mathcal{Z} : \exists m \in [m_0, m_0+B-1] \text{ such that } l+m \in \mathcal{Z}\}\,.$$

Property (iii) of Lemma 6.2 implies that $\mathcal{Z}'$ is piecewise syndetic. For every $l \in \mathcal{Z}'$, there exists $m \in [m_0, m_0+B-1]$ such that $l+m \in \mathcal{Z}$, so that $\boldsymbol{e}(l,m) \in \mathcal{E}$ and

$$(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}-\boldsymbol{\mu}T_{\boldsymbol{e}(l,m)}} = \frac{(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}}}{\left(T_{\boldsymbol{e}(l,m)+\boldsymbol{k}_l}\boldsymbol{\alpha}\right)^{\boldsymbol{\mu}}} = \frac{(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}}}{\left(T_{\boldsymbol{k}_{l+m}}\boldsymbol{\alpha}\right)^{\boldsymbol{\mu}}} = \frac{\gamma}{\gamma} = 1\,.$$

Hence $l \in \mathcal{Z}_{\boldsymbol{e}(l,m)}$. It follows that

$$\mathcal{Z}' \subset \bigcup_{\boldsymbol{e}\in\mathcal{E}} \mathcal{Z}_{\boldsymbol{e}}\,.$$

Property (ii) of Lemma 6.2 ensures the existence of $\boldsymbol{e} \in \mathcal{E}$ such that $\mathcal{Z}_{\boldsymbol{e}}$ is piecewise syndetic. Fix such a vector $\boldsymbol{e} \in \mathcal{E}$. It follows from Lemma 6.6 that $\boldsymbol{\mu} - \boldsymbol{\mu}T_{\boldsymbol{e}} = 0$. Since the coordinates of $\boldsymbol{e}$ are nonzero, we obtain a contradiction with the assumption that none of the eigenvalues of T_i is a root of unity. □

We are now ready to prove Theorem 6.4.

Proof of Theorem 6.4. Set

$$\mathcal{Z} := \{l \in \mathbb{N} : g(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\} \, .$$

Let $\mathbb{K}$ be a number field containing the coordinates of $\boldsymbol{\alpha}$ and $\mathsf{M}_{\mathbb{K}}$ be the set of places of $\mathbb{K}$. The assumption that the pairs $(T_i, \boldsymbol{\alpha}_i)$ are admissible allows us to apply [24, Theorem 3] to the sequence of points $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$. In order to apply the result of Corvaja and Zannier, we need to prove that the following three conditions are satisfied.

(i) There exists a finite set $\mathcal{S} \subset \mathsf{M}_{\mathbb{K}}$ such that the coordinates of the algebraic points $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$ are $\mathcal{S}$-units.
(ii) The sequence $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ tends to 0.
(iii) One has $\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(-\log \|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\|)$, where H is the absolute Weil height defined in Section 4.5.

Condition (i) is easy to check. We recall that, given a finite set of places $\mathcal{S}$ of $\mathbb{K}$, an algebraic number β is a $\mathcal{S}$-unit if $|\beta|_{\mathsf{v}} = 1$ for all $\mathsf{v} \in \mathsf{M}_{\mathbb{K}} \setminus \mathcal{S}$. Any $\beta \in \mathbb{K}^\star$ is a $\mathcal{S}$-unit for some finite set $\mathcal{S} \subset \mathsf{M}_{\mathbb{K}}$ and hence the elements of any finite set of nonzero elements of $\mathbb{K}$ are $\mathcal{S}$-units for some finite set $\mathcal{S} \subset \mathsf{M}_{\mathbb{K}}$. Thus, there exist a finite set $\mathcal{S}_0 \subset \mathsf{M}_{\mathbb{K}}$ such that all the coordinates of the vector $\boldsymbol{\alpha}$ are $\mathcal{S}_0$-units. Since the set of $\mathcal{S}_0$-units is a multiplicative group, the coordinates of $T_{\boldsymbol{k}_l}\boldsymbol{\alpha}$ are $\mathcal{S}_0$-units too for all $l \in \mathbb{N}$.

Since by assumption the pairs $(T_i, \boldsymbol{\alpha}_i)$ are admissible, Theorem 5.10 implies that the sequence $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ tends to 0, and thus (ii) is satisfied.

Now, let us check that (iii) holds. We infer from Lemma 6.5 that

$$\|T_{\boldsymbol{k}_l}\| = \mathcal{O}(e^l) \quad \text{and} \quad \log \|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\| \leq -ce^l \, ,$$

for some positive real number c and all sufficiently large integers l. On the other hand, an easy computation using basic properties of the Weil height gives that $\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(\|T_{\boldsymbol{k}_l}\|)$. It thus follows that

$$\log H(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = \mathcal{O}(-\log \|T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\|) \, ,$$

which shows that Condition (iii) is satisfied.

Applying [24, Theorem 3] to the sequence of algebraic points $(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})_{l\in\mathbb{N}}$ and to the function $g(\boldsymbol{z})$, we obtain the existence of a finite number of nonzero N-tuples of integers $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_s$ and of nonzero algebraic numbers $\gamma_1, \ldots, \gamma_s$, such that

$$\mathcal{Z} \subset \bigcup_{i=1}^{s} \mathcal{Z}_i$$

where

$$\mathcal{Z}_i := \{l \in \mathbb{N} : (T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^{\boldsymbol{\mu}_i} = \gamma_i\} \, .$$

By Lemma 6.7, the sets $\mathcal{Z}_i$ are all negligible. It thus follows from Properties (i) and (ii) of Lemma 6.3 that $\mathcal{Z}$ is also negligible, which completes the proof. □

6.3. **Existence of good sequences** $(k_l)_{l\in\mathbb{N}}$**.** Theorem 6.4 can be applied to any sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset \mathbb{N}^r$ satisfying the asymptotic

$$\boldsymbol{k}_l = \Theta l + \mathcal{O}(1) \, , \tag{6.9}$$

where Θ is defined as in (6.1). However, in order to prove our main result (Theorem 7.2), we will have to choose a sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset \mathbb{N}^r$ satisfying some additional properties. Lemma 6.8 will ensure that sequences with such properties do exist.

Let V denote the orthogonal complement to the vector Θ in $\mathbb{Z}^r$. That is,

$$V := \{\boldsymbol{\mu} \in \mathbb{Z}^r \,:\, \langle \boldsymbol{\mu}, \Theta\rangle = 0\} .$$

Let $V^\perp$ denote the orthogonal complement to V in $\mathbb{Z}^r$. That is,

$$V^\perp := \{\boldsymbol{k} \in \mathbb{Z}^r \,:\, \langle \boldsymbol{k}, \boldsymbol{\mu}\rangle = 0 \text{ for all } \boldsymbol{\mu} \in V\} .$$

We also set

$$V_+^\perp := V^\perp \cap \mathbb{N}^r . \tag{6.10}$$

Recall that the notation $\boldsymbol{k}_1 \leq \boldsymbol{k}_2$ means that the vector $\boldsymbol{k}_2 - \boldsymbol{k}_1$ has nonnegative coordinates.

Lemma 6.8. *We continue with the previous notation. There exists a sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ with values in $\mathbb{N}^r$ satisfying the three following conditions.*

(i) $\boldsymbol{k}_l \in V_+^\perp\,, \quad \forall l \in \mathbb{N}$.

(ii) $\boldsymbol{k}_l = \Theta l + \mathcal{O}(1)$.

(iii) $\boldsymbol{k}_l \leq \boldsymbol{k}_{l+1}\,, \quad \forall l \in \mathbb{N}$.

Proof. We first note that the set V could possibly be reduced to $\{0\}$; this is the case when the numbers

$$1/\log\rho(T_1), \ldots, 1/\log\rho(T_r)$$

are linearly independent over the rational numbers[5]. In that case, one can simply choose

$$\boldsymbol{k}_l := \left(\left\lfloor \frac{l}{\log\rho(T_1)}\right\rfloor, \ldots, \left\lfloor \frac{l}{\log\rho(T_r)}\right\rfloor\right), \quad \forall l \in \mathbb{N} .$$

In contrast, the set $V^\perp$ is a nonzero $\mathbb{Z}$-module. Indeed, the $\mathbb{R}$-vector space generated by $V^\perp$ in $\mathbb{R}^r$ contains the vector Θ. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s$ be a $\mathbb{Z}$-basis of $V^\perp$. There exist real numbers $\lambda_1, \ldots, \lambda_s$ such that

$$\Theta = \lambda_1 \boldsymbol{e}_1 + \cdots + \lambda_s \boldsymbol{e}_s .$$

We deduce that

$$\|l\Theta - \lfloor l\lambda_1\rfloor \boldsymbol{e}_1 + \cdots + \lfloor l\lambda_s\rfloor \boldsymbol{e}_s\| \leq \sum_{i=1}^{s} \|\boldsymbol{e}_i\| . \tag{6.11}$$

Since all coordinates of Θ are positive, there exists a nonnegative integer l_0 such that, for all $l \geq l_0$, the vector $\lfloor l\lambda_1\rfloor \boldsymbol{e}_1 + \cdots + \lfloor l\lambda_s\rfloor \boldsymbol{e}_s$ has positive coordinates. For every $l \geq l_0$, set

$$\boldsymbol{a}_l := \lfloor l\lambda_1\rfloor \boldsymbol{e}_1 + \cdots + \lfloor l\lambda_s\rfloor \boldsymbol{e}_s \in V_+^\perp .$$

The sequence $(\boldsymbol{a}_l)_{l\geq l_0}$ agrees with the asymptotic (ii), but not necessarily with the partial order $\leq$. Let $e := \sum_{i=1}^s \|\boldsymbol{e}_i\|$ and let $\theta > 0$ denote the minimum of the coordinates of the vector Θ. If $l_1 \geq l_0$ and $l_2 \geq l_1 + 2e/\theta$,

[5]When $r > 2$, it is not known whether the pairwise multiplicative independence of the numbers $\rho(T_i)$ implies that the reciprocals of their logarithms are linearly independent over $\mathbb{Q}$.

then (6.11) implies that $\boldsymbol{a}_{l_1} \leq \boldsymbol{a}_{l_2}$. Set $b := \lceil 2e/\theta \rceil$. Let us define the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}} \subset \mathbb{N}^r$ by setting

$$\boldsymbol{k}_{l_0+lb+j} := \boldsymbol{a}_{l_0+lb}$$

for $l \in \mathbb{N}$ and $0 \leq j < b$, and $\boldsymbol{k}_l = \boldsymbol{0}$ for $l < l_0$. Then the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ has all the required properties. □

7. Mahler's method in families

In this section, we state Theorem 7.2, a general lifting theorem dealing with families of Mahler systems associated with sufficiently independent transformations. Sections 8 and 9 will be devoted to the proof of this result. In Section 10, Theorems 3.3, 3.6, and 3.8, as well as Corollary 3.9, will be deduced from this result.

7.1. Statement of Theorem 7.2.

Let r be a positive integer. For every i, $1 \leq i \leq r$, let us consider a Mahler system

$$\begin{pmatrix} f_{i,1}(\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(\boldsymbol{z}_i) \end{pmatrix} = A_i(\boldsymbol{z}_i) \begin{pmatrix} f_{i,1}(T_i\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(T_i\boldsymbol{z}_i) \end{pmatrix} \tag{7.1.i}$$

where n_i and m_i are positive integers, $\boldsymbol{z}_i = (z_{i,1}, \ldots, z_{i,n_i})$ is a vector of indeterminates, $T_i \in \mathcal{M}_{n_i}(\mathbb{N})$ with spectral radius $\rho(T_i)$, $A_i(\boldsymbol{z}_i)$ belongs to $\mathrm{GL}_{m_i}(\overline{\mathbb{Q}}(\boldsymbol{z}_i))$, and $f_{i,1}(\boldsymbol{z}_i), \ldots, f_{i,m_i}(\boldsymbol{z}_i)$ belong to $\overline{\mathbb{Q}}\{\boldsymbol{z}_i\}$. We also let $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^\star)^{n_i}$ and $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,m_i})$ denote a vector of indeterminates. Set $\boldsymbol{z} := (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r)$ and $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$.

Remark 7.1. Note that one has to replace $A_i(\boldsymbol{z}_i)$ by $A_i(\boldsymbol{z}_i)^{-1}$ to obtain a system as in (3.1). However, it is more natural in our proof to work with systems written in the form (7.1.i). We recall that the notation $\overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$ stands for the algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ in $\overline{\mathbb{Q}}\{\boldsymbol{z}-\boldsymbol{\alpha}\}$.

Theorem 7.2. *We continue with the above notation and assumptions. Let us assume that the two following conditions hold.*

(i) *For every i, $\boldsymbol{\alpha}_i$ is regular w.r.t. (7.1.i) and $(T_i, \boldsymbol{\alpha}_i)$ is admissible.*
(ii) *$\rho(T_1), \ldots, \rho(T_r)$ are pairwise multiplicatively independent.*

Then for every polynomial $P \in \overline{\mathbb{Q}}[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$ that is homogeneous with respect to each family of indeterminates $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r$, and such that

$$P(f_{1,1}(\boldsymbol{\alpha}_1), \ldots, f_{r,m_r}(\boldsymbol{\alpha}_r)) = 0\,,$$

there exists a polynomial $Q \in \overline{\overline{\mathbb{Q}}(\boldsymbol{z})}_{\boldsymbol{\alpha}}[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$, homogeneous with respect to each family of indeterminates $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r$, and such that

$$Q(\boldsymbol{z}, f_{1,1}(\boldsymbol{z}_1), \ldots, f_{r,m_r}(\boldsymbol{z}_r)) = 0 \text{ and } Q(\boldsymbol{\alpha}, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_r) = P(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)\,.$$

Furthermore, if $\overline{\mathbb{Q}}(\boldsymbol{z})(f_{1,1}(\boldsymbol{z}_1), \ldots, f_{r,m_r}(\boldsymbol{z}_r))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$, then there exists such a polynomial Q in $\overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$.

7.2. **Notation.** We fix now some additional notation that will be used in the proof of Theorem 7.2, that is all along Sections 8 and 9.

From now on, we assume that the following data from Theorem 7.2 are fixed: the Mahler systems (7.1.i), the points $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^\star)^{n_i}$, $1 \leq i \leq r$ (and thus the point $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$), and a polynomial $P_\star \in \overline{\mathbb{Q}}[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$ such that

$$P_\star(f_{1,1}(\boldsymbol{\alpha}_1), \ldots, f_{r,m_r}(\boldsymbol{\alpha}_r)) = 0\,. \tag{7.2}$$

We set

$$M := \sum_{i=1}^{r} m_i \qquad \text{and} \qquad N := \sum_{i=1}^{r} n_i. \tag{7.3}$$

We also set

$$\boldsymbol{f}_i(\boldsymbol{z}_i) := {}^{\mathrm{t}}(f_{i,1}(\boldsymbol{z}_i), \ldots, f_{i,m_i}(\boldsymbol{z}_i))\,.$$

Iterating k times the system (7.1.i), one obtains the new system

$$\boldsymbol{f}_i(\boldsymbol{z}_i) = A_{i,k}(\boldsymbol{z}_i)\boldsymbol{f}_i(T_i^k \boldsymbol{z}_i)\,, \tag{7.4.i}$$

where

$$A_{i,k}(\boldsymbol{z}_i) := A_i(\boldsymbol{z}_i)A_i(T_i\boldsymbol{z}_i)\cdots A_i(T_i^{k-1}\boldsymbol{z}_i) \quad \forall k \geq 1\,,$$

and $A_{i,0} := I_{m_i}$. By abuse of notation, we set $\boldsymbol{f}_i(\boldsymbol{z}) := \boldsymbol{f}_i(\boldsymbol{z}_i)$ where $\boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r)$. For every r-tuple of positive integers $\boldsymbol{k} = (k_1, \ldots, k_r)$, one can gather the systems (7.4.i) into a single one as follows:

$$\begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_r(\boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} A_{1,k_1}(\boldsymbol{z}_1) & & \\ & \ddots & \\ & & A_{r,k_r}(\boldsymbol{z}_r) \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1(T_{\boldsymbol{k}}\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_r(T_{\boldsymbol{k}}\boldsymbol{z}) \end{pmatrix}, \tag{7.5}$$

where $T_{\boldsymbol{k}} := T_1^{k_1} \oplus \cdots \oplus T_r^{k_r}$. Finally, we set

$$\boldsymbol{f}(\boldsymbol{z}) := {}^{\mathrm{t}}(\boldsymbol{f}_1(\boldsymbol{z}), \ldots, \boldsymbol{f}_r(\boldsymbol{z}))$$

and we let $A_{\boldsymbol{k}}(\boldsymbol{z})$ denote the block diagonal matrix defined so that (7.5) can be shortened to

$$\boldsymbol{f}(\boldsymbol{z}) = A_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{f}(T_{\boldsymbol{k}}\boldsymbol{z})\,. \tag{7.6}$$

7.2.1. *The matrices $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$.* For every i, $1 \leq i \leq r$, we let d_i denote the total degree of $P_\star$ with respect to the indeterminates $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,m_i})$. Let t denote the number of distinct vectors $(\mu_{1,1}, \ldots, \mu_{1,m_1}, \mu_{2,1}, \ldots, \mu_{r,m_r}) \in \mathbb{N}^M$ such that

$$\mu_{i,1} + \cdots + \mu_{i,m_i} = d_i\,, \quad \forall i,\ 1 \leq i \leq r\,.$$

We let $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_t$ denote an enumeration of these vectors. For every i, $1 \leq i \leq r$, let B_i denote an $m_i \times m_i$ matrix with coefficients in some commutative ring $\mathcal{R}$, and set $B := B_1 \oplus \cdots \oplus B_r$. According to the notation of Section 4.1.4, given a column vector of M indeterminates $\boldsymbol{X} := {}^{\mathrm{t}}(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$, we note that $(B\boldsymbol{X})^{\boldsymbol{\mu}_j} \in \mathcal{R}[\boldsymbol{X}]$ is a homogeneous polynomial of degree d_i in each set of variables $\boldsymbol{X}_i$. We let $R_{j,l}(B)$ denote the elements of $\mathcal{R}$ defined by

$$(B\boldsymbol{X})^{\boldsymbol{\mu}_j} = \sum_{l=1}^{t} R_{j,l}(B)\boldsymbol{X}^{\boldsymbol{\mu}_l}\,. \tag{7.7}$$

We also set $R(B) := (R_{j,l}(B))_{1\leq j,l\leq t}$. Let C be another $M \times M$ block diagonal matrix. Then, it follows from (7.7) that, for every j, $1 \leq j \leq t$,

$$\sum_{l=1}^{t} R_{j,l}(BC)\boldsymbol{X}^{\boldsymbol{\mu}_l} = (BC\boldsymbol{X})^{\boldsymbol{\mu}_j} = \sum_{k=1}^{t} R_{j,k}(B)(C\boldsymbol{X})^{\boldsymbol{\mu}_k} = \sum_{l=1}^{t}\sum_{k=1}^{t} R_{j,k}(B)R_{k,l}(C)\boldsymbol{X}^{\boldsymbol{\mu}_l}\,.$$

Thus,

$$R(BC) = R(B)R(C) \tag{7.8}$$

and, in particular,

$$R(B)^{-1} = R(B^{-1}) \tag{7.9}$$

when B is invertible.

Using the previous notation, we define, for every $\boldsymbol{k} \in \mathbb{N}^r$, the matrix

$$\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) := R(A_{\boldsymbol{k}}(\boldsymbol{z})) \in \mathcal{M}_t(\overline{\mathbb{Q}}(\boldsymbol{z}))\,. \tag{7.10}$$

One has $\boldsymbol{R}_{\boldsymbol{0}}(\boldsymbol{z}) = I_t$ and, given $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{N}^r$, it follows from (7.8) that

$$\boldsymbol{R}_{\boldsymbol{k}+\boldsymbol{k}'}(\boldsymbol{z}) = \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}'}(T_{\boldsymbol{k}}\boldsymbol{z})\,. \tag{7.11}$$

Lemma 7.3. *The matrix $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{\alpha})$ is well-defined and invertible for all $\boldsymbol{k} \in \mathbb{N}^r$.*

Proof. Let $\boldsymbol{k} \in \mathbb{N}^r$. Since, for each i, the point $\boldsymbol{\alpha}_i$ is assumed to be regular with respect to A_i, the matrix $A_{\boldsymbol{k}}$ is well-defined and invertible at $\boldsymbol{\alpha}$. We thus infer from (7.9) and (7.10) that $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{\alpha})$ is well-defined and invertible, with inverse $R(A_{\boldsymbol{k}}(\boldsymbol{\alpha})^{-1})$. □

7.2.2. *Choice of the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$.* Since the Mahler systems (7.1.i), $1 \leq i \leq r$, have been fixed, the associated transformations $T_1, \ldots, T_r$ are fixed too. As in Section 6, we define the vector

$$\Theta := \left(\frac{1}{\log \rho(T_1)}, \ldots, \frac{1}{\log \rho(T_r)}\right)$$

and then the corresponding set $V_+^\perp$ (cf. Section 6.3).

Definition 7.4. We define $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ as a fixed sequence in $\mathbb{N}^r$ satisfying the three following conditions.

(i) $\boldsymbol{k}_l \in V_+^\perp\,, \quad \forall l \in \mathbb{N}$.

(ii) $\boldsymbol{k}_l = \Theta l + \mathcal{O}(1)$.

(iii) $\boldsymbol{k}_l \leq \boldsymbol{k}_{l+1}\,, \quad \forall l \in \mathbb{N}$.

By Lemma 6.8, sequences satisfying the properties of Definition 7.4 do exist, so that Definition 7.4 makes sense. This sequence will remain fixed all along Sections 8 and 9.

8. Hilbert's Nullstellensatz and relation matrices

In this section, we gather some preliminary results needed for proving Theorem 7.2. In particular, we introduce the so-called *relation matrices* and study some of their properties.

Let $\boldsymbol{Y} = (y_{i,j})_{1\leq i,j\leq t}$ denote a matrix of indeterminates. Given a field $\mathbb{K}$ and a nonnegative integer δ_1, we let $\mathbb{K}[\boldsymbol{Y}]_{\delta_1}$ denote the set of polynomials of degree at most δ_1 in every indeterminate $y_{i,j}$. Given two nonnegative integers δ_1 and δ_2, we let $\mathbb{K}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$ denote the set of polynomials $P \in \mathbb{K}[\boldsymbol{Y},\boldsymbol{z}]$ of degree at most δ_1 in every indeterminate $y_{i,j}$ and of total degree at most δ_2 in the indeterminates $z_{i,j}$.

Let $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$ be the sequence of Definition 7.4. By Theorem 6.4, every polynomial $P(\boldsymbol{Y},\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ is well-defined at the point $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all l in a full subset of $\mathbb{N}$. We recall that piecewise syndetic, negligible, and full sets are introduced in Definition 6.1.

Remark 8.1. In addition to Lemmas 6.2 and 6.3, we will also use the following simple results. If $\mathcal{E}_1$ is full and $\mathcal{E}_2$ is picewise syndetic, then $\mathcal{E}_1\cap\mathcal{E}_2$ is piecewise syndetic. Indeed we can express $\mathcal{E}_2$ as $(\mathcal{E}_1\cap\mathcal{E}_2)\cup((\mathbb{N}\setminus\mathcal{E}_1)\cap\mathcal{E}_2)$ and, since $\mathcal{E}_2$ is piecewise syndetic, Property (ii) of Lemma 6.2 implies that at least one of the set $\mathcal{E}_1\cap\mathcal{E}_2$ and $(\mathbb{N}\setminus\mathcal{E}_1)\cap\mathcal{E}_2$ is piecewise syndetic. Since $\mathcal{E}_1$ is full, $\mathbb{N}\setminus\mathcal{E}_1$ is negligible, and we conclude that $\mathcal{E}_1\cap\mathcal{E}_2$ is piecewise syndetic. A similar argument shows that if $\mathcal{E}_1$ is negligible and $\mathcal{E}_2$ is piecewise syndetic, then $\mathcal{E}_2\setminus\mathcal{E}_1$ is piecewise syndetic.

8.1. Definition of the ideal $\mathcal{I}$.

The aim of this section is to introduce the ideal $\mathcal{I}$, which will play a central role in the proof of Theorem 7.2.

Since the set $\mathcal{S}$ of polynomials $P \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ that does not vanish at any of the points $T_{\boldsymbol{k}}\boldsymbol{\alpha}$, $\boldsymbol{k}\in\mathbb{N}^r$, is multiplicatively closed, the localization of $\overline{\mathbb{Q}}[\boldsymbol{z}]$ at $\mathcal{S}$ is a Noetherian ring, say $\mathcal{A} := \mathcal{S}^{-1}\overline{\mathbb{Q}}[\boldsymbol{z}]$ (see, for instance, [34, Proposition 1.6, p. 415]). The ring $\mathcal{A}$ is the subring of $\overline{\mathbb{Q}}(\boldsymbol{z})$ made of the rational functions which are well-defined at $T_{\boldsymbol{k}}\boldsymbol{\alpha}$ for all $\boldsymbol{k}\in\mathbb{N}^r$. Since by assumption each of the points $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_r$ is regular with respect to the corresponding Mahler system (7.1.i), the coefficients of the matrices $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$, $\boldsymbol{k}\in\mathbb{N}^r$, belong to the ring $\mathcal{A}$. With any set $\mathcal{Z}\subset\mathbb{N}$, we associate the set of polynomials

$$\mathcal{I}_{\mathcal{Z}} := \{P \in \mathcal{A}[\boldsymbol{Y}] \,:\, P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,,\ \forall l \in \mathcal{Z}\}\ .$$

Lemma 8.2. *For all $\mathcal{Z}\subset\mathbb{N}$, $\mathcal{I}_{\mathcal{Z}}$ is a radical ideal of $\mathcal{A}[\boldsymbol{Y}]$.*

Proof. Let $P_1, P_2 \in \mathcal{I}_{\mathcal{Z}}$. Then P_1+P_2 vanishes at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all $l\in\mathcal{Z}$. Hence $P_1+P_2\in\mathcal{I}_{\mathcal{Z}}$. Let $P_1\in\mathcal{I}_{\mathcal{Z}}$ and $P_2\in\mathcal{A}[\boldsymbol{Y}]$. Then $P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ for all $l\in\mathcal{Z}$. On the other hand, by definition of $\mathcal{A}$, $P_2(\boldsymbol{Y},\boldsymbol{z})$ has no pole at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$, $l\in\mathbb{N}$. It follows that P_1P_2 vanishes at $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ for all $l\in\mathcal{Z}$. Hence $P_1P_2\in\mathcal{I}_{\mathcal{Z}}$, which proves that $\mathcal{I}_{\mathcal{Z}}$ is an ideal.

Let $P\in\mathcal{A}[\boldsymbol{Y}]$ be such that $P^r\in\mathcal{I}_{\mathcal{Z}}$. If l is a nonnegative integer such that $P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})^r = 0$, then $P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$. Thus $P\in\mathcal{I}_{\mathcal{Z}}$ and $\mathcal{I}_{\mathcal{Z}}$ is radical, which proves Lemma 8.2. □

To define the ideal $\mathcal{I}$, we first need to choose a minimal piecewise syndetic set $\mathcal{Z}_0$, and then $\mathcal{I}$ is obtained from $\mathcal{I}_{\mathcal{Z}_0}$ by extension of scalars from $\mathcal{A}$ to $\overline{\mathbb{Q}}(\boldsymbol{z})$.

Definition 8.3. A piecewise syndetic set $\mathcal{Z} \subset \mathbb{N}$ is minimal if, for every piecewise syndetic set $\mathcal{Z}' \subset \mathcal{Z}$, we have $\mathcal{I}_{\mathcal{Z}'} = \mathcal{I}_{\mathcal{Z}}$.

Since $\mathcal{A}$ is Noetherian, $\mathcal{A}[\boldsymbol{Y}]$ is Noetherian too and any increasing sequence of ideals is stationary. This implies the existence of some minimal piecewise syndetic sets. From now on, we fix such a set $\mathcal{Z}_0$ until the end of the proof of Theorem 7.2, and we define $\mathcal{I} := \mathcal{I}_{\mathcal{Z}_0} \otimes_{\mathcal{A}} \overline{\mathbb{Q}}(\boldsymbol{z})$, that is,

$$\mathcal{I} := \{P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}] \ : \ \exists a \in \overline{\mathbb{Q}}[\boldsymbol{z}]\backslash\{0\} \text{ such that } a(\boldsymbol{z})P(\boldsymbol{Y},\boldsymbol{z}) \in \mathcal{I}_{\mathcal{Z}_0}\}\,. \tag{8.1}$$

Note that $\mathcal{I}_{\mathcal{Z}_0} \subset \mathcal{I}$, and by the definition of $\mathcal{I}_{\mathcal{Z}_0}$, we also have

$$a(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,, \quad \forall l \in \mathcal{Z}_0\,.$$

Lemma 8.4. *Let $P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ and set*

$$\mathcal{Z}(P) := \{l \in \ \mathcal{Z}_0 \ : \ P(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\}\ .$$

Then the following properties hold.

(i) *If $P \in \mathcal{I}$, then $\mathcal{Z}_0 \setminus \mathcal{Z}(P)$ is negligible.*
(ii) *If $P \in \mathcal{I}$ and $a \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$ is such that $aP \in \mathcal{A}[\boldsymbol{Y}]$, then $aP \in \mathcal{I}_{\mathcal{Z}_0}$.*
(iii) *If $P \notin \mathcal{I}$, then $\mathcal{Z}(P)$ is negligible.*

Proof. Let $P \in \mathcal{I}$. Then there exists $a(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$ such that $aP \in \mathcal{I}_{\mathcal{Z}_0}$. By Theorem 6.4 and Property (i) of Lemma 6.3, we conclude that $\mathcal{Z}(a) = \{l \in \mathcal{Z}_0 \ : \ a(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\}$ is negligible. On the other hand, the definition of $\mathcal{I}_{\mathcal{Z}_0}$ implies that $\mathcal{Z}_0 \setminus \mathcal{Z}(P) \subset \mathcal{Z}(a)$. Using Property (i) of Lemma 6.3, we deduce that $\mathcal{Z}_0 \setminus \mathcal{Z}(P)$ is negligible, which proves (i).

Next, let $P \in \mathcal{I}$ and assume that there exists $a \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$ such that $aP \in \mathcal{A}[\boldsymbol{Y}]$. Then, for every $l \in \mathcal{Z}(P)$, we have

$$a(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})P(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,,$$

that is $a(\boldsymbol{z})P(\boldsymbol{Y},\boldsymbol{z}) \in \mathcal{I}_{\mathcal{Z}(P)}$. Since $\mathcal{Z}_0$ can be expressed as $\mathcal{Z}(P) \cup (\mathcal{Z}_0 \setminus \mathcal{Z}(P))$ and $\mathcal{Z}_0 \setminus \mathcal{Z}(P)$ is negligible, it follows from Property (ii) of Lemma 6.2 that $\mathcal{Z}(P)$ is piecewise syndetic. By minimality of $\mathcal{Z}_0$, we deduce that $a(\boldsymbol{z})P(\boldsymbol{Y},\boldsymbol{z}) \in \mathcal{I}_{\mathcal{Z}_0}$, which proves (ii).

Finally, to prove (iii), we proceed by contraposition. Suppose that $\mathcal{Z}(P)$ is not negligible and is therefore piecewise syndetic. By minimality of $\mathcal{Z}_0$, we have $\mathcal{I}_{\mathcal{Z}(P)} = \mathcal{I}_{\mathcal{Z}_0}$. By definition of $\mathcal{I}_{\mathcal{Z}(P)}$, we know that $aP \in \mathcal{I}_{\mathcal{Z}(P)} = \mathcal{I}_{\mathcal{Z}_0}$ for some $a \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$. Thus, $P \in \mathcal{I}$,which proves (iii). ☐

We will now prove the following lemma.

Lemma 8.5. *The ideal $\mathcal{I}$ is a radical ideal of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$.*

Proof. Let $P_1, P_2 \in \mathcal{I}$. Then there exist $a_1, a_2 \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$ such that both a_1P_1 and a_2P_2 belong to $\mathcal{A}[\boldsymbol{Y}]$. By Lemma 8.4, we have that a_1P_1 and aP_2 belong to $\mathcal{I}_{\mathcal{Z}_0}$. By Lemma 8.2, $a_1a_2(P_1 + P_2) \in \mathcal{I}_{\mathcal{Z}_0}$, and thus $P_1 + P_2 \in \mathcal{I}$. Next, let $P_1 \in \mathcal{I}$ and $P_2 \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$. Then there exist $a_1, a_2 \in \overline{\mathbb{Q}}[\boldsymbol{z}]\setminus\{0\}$ such that $a_1P_1 \in \mathcal{I}_{\mathcal{Z}_0}$ and $a_2P_2 \in \mathcal{A}[\boldsymbol{Y}]$. Since $\mathcal{I}_{\mathcal{Z}_0}$ is an ideal, $a_1a_2(P_1P_2) \in \mathcal{I}_{\mathcal{Z}_0}$, and thus $P_1P_2 \in \mathcal{I}$. Finally, let $P \in \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$ be such that $P^r \in \mathcal{I}$ for some

r. Let $a \in \overline{\mathbb{Q}}[\boldsymbol{z}] \setminus \{0\}$ be such that $aP \in \overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{Y}]$. Then $a^r P^r \in \mathcal{A}[\boldsymbol{Y}]$ and, since $P^r \in \mathcal{I}$, it follows from (ii) of Lemma 8.4 that $a^r P^r = (aP)^r \in \mathcal{I}_{\mathcal{Z}_0}$. By Lemma 8.2, we conclude that $aP \in \mathcal{I}_{\mathcal{Z}_0}$. Hence, $P \in \mathcal{I}$ and thus $\mathcal{I}$ is a radical ideal. □

8.2. **Estimates for the dimension of certain vector spaces.** Let δ_1 and δ_2 be two nonnegative integers. Set $\mathcal{I}(\delta_1) := \mathcal{I} \cap \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]_{\delta_1}$ and $\mathcal{I}(\delta_1, \delta_2) := \mathcal{I} \cap \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$. Note that $\mathcal{I}(\delta_1, \delta_2)$ is a $\overline{\mathbb{Q}}$-vector subspace of $\overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$. Let $\mathcal{I}^c(\delta_1, \delta_2)$ denote a complement vector space in $\overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2}$, that is

$$\mathcal{I}(\delta_1, \delta_2) \oplus \mathcal{I}^c(\delta_1, \delta_2) = \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{\delta_1,\delta_2} .$$

We recall that $N = n_1 + \cdots + n_r$ is the number of distinct variables $z_{i,j}$ which form the coordinates of $\boldsymbol{z}$.

Lemma 8.6. *Let $d(\delta_1, \delta_2)$ denote the dimension of $\mathcal{I}^c(\delta_1, \delta_2)$. There exists a positive real number $c_1(\delta_1)$, that does not depend on δ_2, such that*

$$d(\delta_1, \delta_2) \sim c_1(\delta_1)\delta_2^N , \text{ as } \delta_2 \text{ tends to infinity.}$$

Proof. Let $h := (\delta_1 + 1)^{t^2}$ denote the number of distinct monomials of degree at most δ_1 in every indeterminates $y_{i,j}$ and let $\boldsymbol{Y}^{\boldsymbol{\nu}_1}, \ldots, \boldsymbol{Y}^{\boldsymbol{\nu}_h}$ denote an enumeration of these monomials. Let $P_1, \ldots, P_h$ be polynomials in $\mathcal{I}(\delta_1)$. Every P_i has a unique decomposition of the form

$$P_i(\boldsymbol{Y}, \boldsymbol{z}) = \sum_{j=1}^{h} p_{i,j}(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j} ,$$

where $p_{i,j}(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$, $1 \le i, j \le h$. Consider the square matrix $C(\boldsymbol{z}) := (p_{i,j}(\boldsymbol{z}))_{1\le i,j\le h}$. By Theorem 6.4, the set

$$\mathcal{E}_0 := \{l \in \mathcal{Z}_0 : C(\boldsymbol{z}) \text{ has a pole at } T_{\boldsymbol{k}_l}\boldsymbol{\alpha}\}$$

is negligible. For every i, $1 \le i \le h$, we define the set

$$\mathcal{E}_i := \{l \in \mathcal{Z}_0 : P_i(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0\} .$$

Since $P_1, \ldots, P_h \in \mathcal{I}$, it follows from Lemma 8.4 that the sets $\mathcal{E}_1, \ldots, \mathcal{E}_h$ are all negligible. Property (ii) of Lemma 6.3 then implies that $\mathcal{E} := \bigcup_{i=0}^{h} \mathcal{E}_i$ is also negligible. For every $l \in \mathcal{Z}_0 \setminus \mathcal{E}$, we have

$$C(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_1} \\ \vdots \\ \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_h} \end{pmatrix} = \begin{pmatrix} P_1(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ \vdots \\ P_h(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \end{pmatrix} = 0 . \tag{8.2}$$

By Lemma 7.3, the matrix $\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$ is invertible. In particular it is nonzero. Hence the vector $(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_1}, \ldots, \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})^{\boldsymbol{\nu}_h})$ is also nonzero. From (8.2), we deduce that $\det C(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0$ for all $l \in \mathcal{Z}_0 \setminus \mathcal{E}$. Since $\mathcal{E}$ is negligible, $\mathcal{Z}_0 \setminus \mathcal{E}$ is piecewise syndetic (cf. Remark 8.1). Thus, by Theorem 6.4, we conclude that $\det C(\boldsymbol{z}) = 0$. Hence $\mathcal{I}(\delta_1)$ is a strict subspace of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]_{\delta_1}$, say of dimension $d < h$. Thus, there exist $h - d$ linear forms $\ell_1, \ldots, \ell_{h-d} : \overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]_{\delta_1} \to \overline{\mathbb{Q}}(\boldsymbol{z})$, linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})$ and such that

$$P \in \mathcal{I}(\delta_1) \iff \ell_1(P) = \cdots = \ell_{h-d}(P) = 0 . \tag{8.3}$$

For any i, $1 \le i \le h-d$, we have a decomposition

$$\ell_i\left(\sum_{j=1}^{h} p_j(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j}\right) = \sum_{j=1}^{h} b_{i,j}(\boldsymbol{z})p_j(\boldsymbol{z})\,.$$

Without any loss of generality, we can assume that $b_{i,j}(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$, $1 \le i \le h-d$, $1 \le j \le h$, and we let $c(\delta_1) \ge 0$ denote the maximum of the total degree of the polynomials $b_{i,j}(\boldsymbol{z})$. It follows from (8.3) that, for all $p_1(\boldsymbol{z}),\ldots,p_h(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$,

$$(8.4)\quad \sum_{j=1}^{h} p_j(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \mathcal{I}(\delta_1) \iff \sum_{j=1}^{h} b_{i,j}(\boldsymbol{z})p_j(\boldsymbol{z}) = 0 \quad \forall i,\ 1 \le i \le h-d\,.$$

The fact that $\ell_1,\ldots,\ell_{h-d}$ are linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})$ implies that, for any $q_1(\boldsymbol{z}),\ldots,q_{h-d}(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})$,

$$(8.5)\quad \sum_{i=1}^{h-d} q_i(\boldsymbol{z})b_{i,j}(\boldsymbol{z}) = 0\,,\ \forall j,\ 1 \le j \le h \iff q_i(\boldsymbol{z}) = 0\,,\ \forall i,\ 1 \le i \le h-d\,.$$

Now, let us consider a polynomial $P = \sum_{j=1}^{h} p_j(\boldsymbol{z})\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$. We have decompositions

$$p_j(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\,:\,|\boldsymbol{\lambda}|\le\delta_2} p_{j,\boldsymbol{\lambda}}\boldsymbol{z}^{\boldsymbol{\lambda}} \quad\text{and}\quad b_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{\kappa}\,:\,|\boldsymbol{\kappa}|\le c(\delta_1)} b_{i,j,\boldsymbol{\kappa}}\boldsymbol{z}^{\boldsymbol{\kappa}}\,,$$

where the numbers $p_{j,\boldsymbol{\lambda}}$ and $b_{i,j,\boldsymbol{\kappa}}$ are algebraic for all quadruples $(i,j,\boldsymbol{\lambda},\boldsymbol{\kappa})$. By (8.4), we obtain that

$$(8.6)\quad P \in \mathcal{I}(\delta_1,\delta_2) \iff \sum_{j=1}^{h} \sum_{\substack{|\boldsymbol{\lambda}|\le\delta_2,|\boldsymbol{\kappa}|\le c(\delta_1)\\ \boldsymbol{\lambda}+\boldsymbol{\kappa}=\boldsymbol{\gamma}}} b_{i,j,\boldsymbol{\kappa}}p_{j,\boldsymbol{\lambda}} = 0\,,\quad \forall(i,\boldsymbol{\gamma}),\ 1 \le i \le h-d,\ |\boldsymbol{\gamma}| \le \delta_2 + c(\delta_1)\,.$$

Let $\Gamma(\delta_1,\delta_2)$ denote the set of $\boldsymbol{\gamma} \in \mathbb{N}^N$ whose coordinates are all larger than $c(\delta_1)$ and such that $|\boldsymbol{\gamma}| \le \delta_2$. Let $V(\delta_1,\delta_2)$ be the $\overline{\mathbb{Q}}$-vector subspace of the dual of $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$ spanned by the linear forms

$$(8.7)\quad L_{i,\boldsymbol{\gamma}} : P \mapsto \sum_{j=1}^{h} \sum_{|\boldsymbol{\lambda}|\le\delta_2,} b_{i,j,\boldsymbol{\gamma}-\boldsymbol{\lambda}}p_{j,\boldsymbol{\lambda}}\,,\quad 1 \le i \le h-d\,,\ \boldsymbol{\gamma} \in \Gamma(\delta_1,\delta_2)\,,$$

where we let $b_{i,j,\boldsymbol{\gamma}-\boldsymbol{\lambda}} = 0$ if $\boldsymbol{\gamma}-\boldsymbol{\lambda} \notin \mathbb{N}^N$ or $|\boldsymbol{\gamma}-\boldsymbol{\lambda}| > c(\delta_1)$. We infer from (8.6) and (8.7) that

$$(8.8)\quad \dim V(\delta_1,\delta_2) \le \dim \mathcal{I}^c(\delta_1,\delta_2) \le \dim V(\delta_1,\delta_2) + \mathcal{O}(\delta_2^{N-1})\,,$$

where the underlying constant in $\mathcal{O}$ depends on δ_1 but not on δ_2.

Now, let us estimate the dimension of $V(\delta_1,\delta_2)$. We first prove that all the linear forms $L_{\boldsymbol{\gamma},i}$, $1 \le i \le h-d$, $\boldsymbol{\gamma} \in \Gamma(\delta_1,\delta_2)$ are linearly independent over $\overline{\mathbb{Q}}$. Let $q_{i,\boldsymbol{\gamma}}$ be algebraic numbers such that

$$\sum_{i=1}^{h-d} \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} q_{i,\boldsymbol{\gamma}} L_{i,\boldsymbol{\gamma}} = 0\,.$$

Then, for every j, $1 \leq j \leq h$, and every $\boldsymbol{\lambda}$, $|\boldsymbol{\lambda}| \leq \delta_2$, we have

$$\sum_{i=1}^{h-d} \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} q_{i,\boldsymbol{\gamma}} b_{i,j,\boldsymbol{\gamma}-\boldsymbol{\lambda}} = 0\,.$$

Set $q_i(\boldsymbol{z}) := \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} q_{i,\boldsymbol{\gamma}} \boldsymbol{z}^{-\boldsymbol{\gamma}} \in \overline{\mathbb{Q}}(\boldsymbol{z})$. Then, for every j, $1 \leq j \leq h$, we obtain that

$$\begin{aligned}\sum_{i=1}^{h-d} q_i(\boldsymbol{z}) b_{i,j}(\boldsymbol{z}) &= \sum_{i=1}^{h-d} \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} \sum_{|\boldsymbol{\kappa}|\leq c(\delta_1)} q_{i,\boldsymbol{\gamma}} b_{i,j,\boldsymbol{\kappa}} z^{\boldsymbol{\kappa}-\boldsymbol{\gamma}} \\ &= \sum_{i=1}^{h-d} \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} \sum_{|\boldsymbol{\lambda}|\leq \delta_2} q_{i,\boldsymbol{\gamma}} b_{i,j,\boldsymbol{\gamma}-\boldsymbol{\lambda}} z^{-\boldsymbol{\lambda}} \\ &= \sum_{|\boldsymbol{\lambda}|\leq\delta_2} z^{-\boldsymbol{\lambda}} \left(\sum_{i=1}^{h-d} \sum_{\boldsymbol{\gamma}\in\Gamma(\delta_1,\delta_2)} q_{i,\boldsymbol{\gamma}} b_{i,j,\boldsymbol{\gamma}-\boldsymbol{\lambda}} \right) = 0\,.\end{aligned}$$

It follows from (8.5) that $q_i(\boldsymbol{z}) = 0$ for all i. Hence $q_{i,\boldsymbol{\gamma}} = 0$ for all $(i, \boldsymbol{\gamma})$, which proves that the linear forms $L_{i,\boldsymbol{\gamma}}$ are linearly independent over $\overline{\mathbb{Q}}$. By definition of $V(\delta_1, \delta_2)$ (cf. (8.7)), we obtain that

$$\dim V(\delta_1, \delta_2) = (h - d) \times \mathrm{Card}(\Gamma(\delta_1, \delta_2))\,.$$

Now, since the map $\boldsymbol{\gamma} \mapsto \boldsymbol{\gamma} - (c(\delta_1), \ldots, c(\delta_1))$ induces a bijection between $\Gamma(\delta_1, \delta_2)$ and $\{\boldsymbol{\gamma} : |\boldsymbol{\gamma}| \leq \delta_2 - c(\delta_1)\}$, we have

$$\mathrm{Card}(\Gamma(\delta_1, \delta_2)) = \binom{\delta_2 - c(\delta_1) + N + 1}{N} \sim \frac{1}{N!}\delta_2^N, \quad \text{as } \delta_2 \to \infty\,.$$

We thus obtain that

$$\dim V(\delta_1, \delta_2) \sim c_1(\delta_1)\delta_2^N\,, \tag{8.9}$$

with $c_1(\delta_1) = (h - d)/N! > 0$ (we recall that h and d depend only on δ_1 and that $d < h$). Finally, we infer from (8.8) that

$$\dim \mathcal{I}^c(\delta_1, \delta_2) \sim c_1(\delta_1)\delta_2^N\,,$$

as wanted. This ends the proof. □

Lemma 8.7. *For every pair of nonnegative integers* (δ_1, δ_2)*, one has*

$$\dim \mathcal{I}^c(2\delta_1, \delta_2) \leq 2^{t^2} \dim \mathcal{I}^c(\delta_1, \delta_2)\,.$$

Proof. Let $P(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{z}]_{2\delta_1,\delta_2}$. Such a polynomial has a decomposition of the form

$$P(\boldsymbol{Y}, \boldsymbol{z}) = \sum_{\ell=1}^{t^2} e_\ell(\boldsymbol{Y})^{\delta_1} P_\ell(\boldsymbol{Y}, \boldsymbol{z})\,, \tag{8.10}$$

where we let $e_\ell(\boldsymbol{Y})$ denote the 2^{t^2} monomials of degree at most one in the indeterminates $y_{i,j}$, and where each $P_\ell(\boldsymbol{Y},\boldsymbol{z})$ belongs to $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$. If, in such a decomposition, every polynomial P_ℓ belongs to $\mathcal{I}(\delta_1,\delta_2)$, then P belongs to $\mathcal{I}(2\delta_1,\delta_2)$. The decomposition (8.10) naturally defines a surjective linear map from $\left(\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}/\mathcal{I}(\delta_1,\delta_2)\right)^{2^{t^2}}$ to $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,\delta_2}/\mathcal{I}(2\delta_1,\delta_2)$. It follows that

$$\dim_{\overline{\mathbb{Q}}}\mathcal{I}^c(2\delta_1,\delta_2)\leq 2^{t^2}\dim_{\overline{\mathbb{Q}}}\mathcal{I}^c(\delta_1,\delta_2)\,,$$

as wanted. □

8.3. **Nullstellensatz and relation matrices.** In this section, we show how Hilbert's Nullstellensatz allows us to exhibit a matrix $\boldsymbol{\phi}$, called a *relation matrix*, whose coordinates are all algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$, and which encodes the algebraic relations over $\overline{\mathbb{Q}}(\boldsymbol{z})$ of degree at most d_i in each variables between the functions $f_{1,1}(\boldsymbol{z}),\ldots,f_{r,m_r}(\boldsymbol{z})$. These relation matrices are the cornerstone of the proof of Theorem 7.2.

Let $\mathbb{K}$ denote an algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$.

Lemma 8.8. *There exists a matrix $\boldsymbol{\phi}\in\mathrm{GL}_t(\mathbb{K})$ such that*

$$P(\boldsymbol{\phi},\boldsymbol{z})=0\,,$$

for all polynomials $P\in\mathcal{I}$.

Proof. Let us consider the affine algebraic set $\mathcal{V}$ associated with the radical ideal $\mathcal{I}$. That is,

$$\mathcal{V}:=\{\boldsymbol{\phi}\in\mathcal{M}_t(\mathbb{K})\ :\ P(\boldsymbol{\phi},\boldsymbol{z})=0\,,\ \forall P\in\mathcal{I}\}\,.$$

According to the weak form of Hilbert's Nullstellensatz (see, for instance, [34, Theorem 1.4, p. 379]), $\mathcal{V}$ is nonempty as soon as $\mathcal{I}$ is a proper ideal of $\overline{\mathbb{Q}}(\boldsymbol{z})[\boldsymbol{Y}]$. But the definition of $\mathcal{I}$ clearly implies that nonzero constant polynomials do not belong to $\mathcal{I}$. Hence $\mathcal{V}$ is nonempty.

Now, let us assume by contradiction that $\det\boldsymbol{\phi}=0$ for all $\boldsymbol{\phi}$ in $\mathcal{V}$. By Hilbert's Nullstellensatz (see, for instance, [34, Theorem 1.5, p. 380]), the polynomial $\det\boldsymbol{Y}$ belongs to the radical of the ideal $\mathcal{I}$. Hence $\det\boldsymbol{Y}\in\mathcal{I}$ since $\mathcal{I}$ is a radical ideal. Property (i) of Lemma 8.4 then implies that there exists $l\in\mathcal{Z}_0$ such that $\det\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})=0$. This leads to a contradiction, as by Lemma 7.3, the matrix $\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$ is invertible for all l in $\mathbb{N}$. We thus conclude that there exists an invertible matrix $\boldsymbol{\phi}$ in $\mathcal{V}$, as required. □

Definition 8.9. A matrix $\boldsymbol{\phi}\in\mathrm{GL}_t(\mathbb{K})$ satisfying the property of Lemma 8.8 is called a *relation matrix*.

The next lemma plays a central role in the proof of Theorem 7.2.

Lemma 8.10. *There exists a piecewise syndetic set $\mathcal{Z}_1\subset\mathcal{Z}_0$ such that the following holds: For any relation matrix $\boldsymbol{\phi}\in\mathrm{GL}_t(\mathbb{K})$, any $P\in\mathcal{I}$, and any $l',l\in\mathcal{Z}_1$ with $l'\geq l$, we have*

$$P\left(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}),T_{\boldsymbol{k}}\boldsymbol{z}\right)=0\,,$$

where $\boldsymbol{k}=\boldsymbol{k}_{l'}-\boldsymbol{k}_l$.

Proof. We recall that the set $V_+^\perp$ is defined in Section 7.2.2 (see also Equality (6.10)). The proof is divided into the following five simple results, namely Facts 1 to 5.

Fact 1. There exist infinitely many r-tuples $\boldsymbol{k} \in V_+^\perp$ such that

$$\mathcal{Z}_0(\boldsymbol{k}) := \{l \in \mathcal{Z}_0 \ : \ \exists l' \in \mathcal{Z}_0,\ \boldsymbol{k}_{l'} - \boldsymbol{k}_l = \boldsymbol{k}\} \tag{8.11}$$

is a piecewise syndetic set.

Recall that $\mathcal{Z}_0$ is piecewise syndetic. Let B be a bound for $\mathcal{Z}_0$ and $e \geq 0$ be an integer. We set

$$\mathcal{K}_e := \{\boldsymbol{k}_{l'} - \boldsymbol{k}_l : (l, l') \in \mathcal{Z}_0^2\,,\ l' - l \in [e, e+B-1]\}\,.$$

Since $\boldsymbol{k}_l, \boldsymbol{k}_{l'} \in V_+^\perp$ and $\boldsymbol{k}_l \leq \boldsymbol{k}_{l'}$, $\mathcal{K}_e \subset V_+^\perp$. Furthermore, it follows from Property (ii) of Definition 7.4 that $\mathcal{K}_e$ is finite. Consider the set

$$\mathcal{Z}_e := \{l \in \mathcal{Z}_0 \ : \ \exists l' \in \mathcal{Z}_0\,,\ l' - l \in [e, e+B-1]\}\,.$$

By Property (iii) of Lemma 6.2, $\mathcal{Z}_e$ is piecewise syndetic. Since

$$\mathcal{Z}_e \subset \bigcup_{\boldsymbol{k} \in \mathcal{K}_e} \mathcal{Z}_0(\boldsymbol{k})\,,$$

it follows from Property (ii) of Lemma 6.2 that at least one of the sets $\mathcal{Z}_0(\boldsymbol{k})$, $\boldsymbol{k} \in \mathcal{K}_e$, is piecewise syndetic. When $e' \gg e$, then $\mathcal{K}_{e'} \cap \mathcal{K}_e = \emptyset$. Thus, letting e run through $\mathbb{N}$, we obtain that there exist infinitely many $\boldsymbol{k} \in V_+^\perp$ such that $\mathcal{Z}_0(\boldsymbol{k})$ is piecewise syndetic. This proves Fact 1.

Recall that we let $\mathcal{A}$ denote the subring of $\overline{\mathbb{Q}}(\boldsymbol{z})$ made of rational functions with no poles in the set $\{T_{\boldsymbol{k}_l}\boldsymbol{\alpha} \ : \ l \in \mathbb{N}\}$ (cf. Section 8.1). Given $\boldsymbol{k} \in V_+^\perp$, we define an action from the (additive) monoid $V_+^\perp$ to $\mathcal{A}[\boldsymbol{Y}]$ by:

$$\sigma_{\boldsymbol{k}} : \left\{ \begin{array}{ccc} \mathcal{A}[\boldsymbol{Y}] & \rightarrow & \mathcal{A}[\boldsymbol{Y}] \\ P(\boldsymbol{Y}, \boldsymbol{z}) & \mapsto & P(\boldsymbol{Y}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z})\,. \end{array} \right.$$

Note that the map $\sigma_{\boldsymbol{k}}$ is well-defined. Indeed, we already observed that the coordinates of $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ belong to the ring $\mathcal{A}$ for all $\boldsymbol{k} \in \mathbb{N}^r$. Furthermore, it follows from (7.11) that, for any $\boldsymbol{k}, \boldsymbol{k}' \in V_+^\perp$,

$$\sigma_{\boldsymbol{k}+\boldsymbol{k}'} = \sigma_{\boldsymbol{k}} \circ \sigma_{\boldsymbol{k}'}\,. \tag{8.12}$$

We also let

$$\mathcal{K} := \{\boldsymbol{k} \in V_+^\perp : \ \mathcal{Z}_0(\boldsymbol{k}) \ \text{ is piecewise syndetic}\}\,.$$

Note that if $\boldsymbol{k} \in \mathcal{K}$, then the set $\mathcal{Z}_0(\boldsymbol{k})$ is nonempty and thus the definition of $\mathcal{Z}_0(\boldsymbol{k})$ implies the existence of $l, l' \in \mathcal{Z}_0$ such that $\boldsymbol{k} = \boldsymbol{k}_{l'} - \boldsymbol{k}_l$.

Fact 2. For all $\boldsymbol{k} \in \mathcal{K}$, $\sigma_{\boldsymbol{k}}(\mathcal{I}_{\mathcal{Z}_0}) \subset \mathcal{I}_{\mathcal{Z}_0}$.

Let $P \in \mathcal{I}_{\mathcal{Z}_0}$, $\boldsymbol{k} \in \mathcal{K}$, and $l \in \mathcal{Z}_0(\boldsymbol{k})$. Let $l' \in \mathcal{Z}_0$ be such that $\boldsymbol{k}_{l'} = \boldsymbol{k} + \boldsymbol{k}_l$. Then, we have

$$\begin{array}{rcl} \sigma_{\boldsymbol{k}}(P)(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) & = & P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}), T_{\boldsymbol{k}}T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ & = & P(\boldsymbol{R}_{\boldsymbol{k}_l+\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}+\boldsymbol{k}_l}\boldsymbol{\alpha}) \\ & = & P(\boldsymbol{R}_{\boldsymbol{k}_{l'}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_{l'}}\boldsymbol{\alpha}) \\ & = & 0\,. \end{array}$$

Thus, $\sigma_{\boldsymbol{k}}(P)$ belongs to the ideal $\mathcal{I}_{\mathcal{Z}_0(\boldsymbol{k})}$. Since $\mathcal{Z}_0(\boldsymbol{k})$ is piecewise syndetic, the minimality of $\mathcal{Z}_0$ implies that $\mathcal{I}_{\mathcal{Z}_0(\boldsymbol{k})} = \mathcal{I}_{\mathcal{Z}_0}$, which proves Fact 2.

Fact 3. Let $P \in \mathcal{I}_{\mathcal{Z}_0}$ be such that $P = \sigma_{\boldsymbol{k}}(Q)$ for some $Q \in \mathcal{A}[\boldsymbol{Y}]$ and $\boldsymbol{k} \in \mathcal{K}$. Then $Q \in \mathcal{I}_{\mathcal{Z}_0}$.

Consider a map $\pi : \mathcal{Z}_0(\boldsymbol{k}) \to \mathbb{N}$ which sends every integer $l \in \mathcal{Z}_0(\boldsymbol{k})$ to an integer $l' \in \mathcal{Z}_0$ such that $\boldsymbol{k}_{l'} = \boldsymbol{k}_l + \boldsymbol{k}$. It follows from Property (ii) of Definition 7.4 that $|\pi(l) - l|$ is bounded. Since $\mathcal{Z}_0(\boldsymbol{k})$ is piecewise syndetic, we infer from Property (v) of Lemma 6.2 that the set

$$\pi(\mathcal{Z}_0(\boldsymbol{k})) = \{\pi(l) \ : \ l \in \mathcal{Z}_0(\boldsymbol{k})\}$$

is piecewise syndetic. The minimality of $\mathcal{Z}_0$ implies that $\mathcal{I}_{\pi(\mathcal{Z}_0(\boldsymbol{k}))} = \mathcal{I}_{\mathcal{Z}_0}$. Let $l' \in \pi(\mathcal{Z}_0(\boldsymbol{k}))$ and let $l \in \mathcal{Z}_0(\boldsymbol{k}) \subset \mathcal{Z}_0$ be such that $\boldsymbol{k}_{l'} = \boldsymbol{k}_l + \boldsymbol{k}$. Then

$$\begin{aligned}
Q(\boldsymbol{R}_{\boldsymbol{k}_{l'}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_{l'}}\boldsymbol{\alpha}) &= Q(\boldsymbol{R}_{\boldsymbol{k}_l+\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}+\boldsymbol{k}_l}\boldsymbol{\alpha}) \\
&= Q(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}), T_{\boldsymbol{k}}T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \\
&= \sigma_{\boldsymbol{k}}(Q)(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) \\
&= P(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}(\boldsymbol{\alpha})) \\
&= 0\,.
\end{aligned}$$

Thus, $Q \in \mathcal{I}_{\pi(\mathcal{Z}_0(\boldsymbol{k}))} = \mathcal{I}_{\mathcal{Z}_0}$. This proves Fact 3.

We let $\mathbb{Z}\langle\mathcal{K}\rangle$ denote the $\mathbb{Z}$-module generated by $\mathcal{K}$ in $\mathbb{Z}^r$.

Fact 4. The set $V^{\perp}/\mathbb{Z}\langle\mathcal{K}\rangle$ is finite.

We shall prove that $V^{\perp}/\mathbb{Z}\langle\mathcal{K}\rangle$ is a finitely generated abelian torsion group. Fact 4 will immediately follow from this result. The fact that $V^{\perp}/\mathbb{Z}\langle\mathcal{K}\rangle$ is a finitely generated abelian group is straightforward. It remains only to prove that it is torsion, i.e., that for any $\boldsymbol{w} \in V^{\perp}$, there exists $a \in \mathbb{Z} \setminus \{0\}$ such that $a\boldsymbol{w} \in \mathbb{Z}\langle\mathcal{K}\rangle$.

We let $\mathbb{Q}\langle\mathcal{K}\rangle$, $\mathbb{Q}\langle V\rangle$, and $\mathbb{Q}\langle V^{\perp}\rangle$ denote the $\mathbb{Q}$-vector subspaces of $\mathbb{Q}^r$ generated, respectively, by $\mathcal{K}$, V, and $V^{\perp}$. Note that $\mathbb{Q}\langle V^{\perp}\ \rangle = \mathbb{Q}\langle V\rangle^{\perp}$. We first prove that $\mathbb{Q}\langle\mathcal{K}\rangle = \mathbb{Q}\langle V\rangle^{\perp}$. Since both $\mathbb{Q}\langle\mathcal{K}\rangle$ and $\mathbb{Q}\langle V\rangle$ are finite dimensional $\mathbb{Q}$-vector spaces, it is equivalent to prove that $\mathbb{Q}\langle\mathcal{K}\rangle^{\perp} = \mathbb{Q}\langle V\rangle$. Since $\mathcal{K} \subset V^{\perp}$, we have $\mathbb{Q}\langle\mathcal{K}\rangle \subset \mathbb{Q}\langle V^{\perp}\ \rangle = \mathbb{Q}\langle V\rangle^{\perp}$. Thus, $\mathbb{Q}\langle V\rangle \subset \mathbb{Q}\langle\mathcal{K}\rangle^{\perp}$. Next, we prove the reverse inclusion. Let $\boldsymbol{\lambda} \in \mathbb{Q}\langle\mathcal{K}\rangle^{\perp}$. Then $\boldsymbol{\lambda}$ is orthogonal to all elements $\boldsymbol{k} \in \mathcal{K}$. Renormalizing, we have the following relation:

$$\left\langle \boldsymbol{\lambda}, \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \right\rangle = 0\,, \qquad \text{for all nonzero } \boldsymbol{k} \in \mathcal{K}\,. \tag{8.13}$$

From Fact 1, we know that $\mathcal{K}$ is infinite. Since every element of $\mathcal{K}$ can be expressed as the difference of two elements of the sequence $(\boldsymbol{k}_l)_{l\in\mathbb{N}}$, by Property (ii) of Definition 7.4, all elements of $\mathcal{K}$ remain at a bounded distance from $\mathbb{R}\Theta$. Taking the limit in (8.13) along a sequence of vectors $\boldsymbol{k} \in \mathcal{K}$ whose norm tends to infinity, we deduce that $\boldsymbol{\lambda}$ is orthogonal to Θ. Thus, $\boldsymbol{\lambda} \in \mathbb{Q}\langle V\rangle$. Therefore, we conclude that $\mathbb{Q}\langle\mathcal{K}\rangle = \mathbb{Q}\langle V\rangle^{\perp}$.

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s \in \mathcal{K}$ form a basis of $\mathbb{Z}\langle\mathcal{K}\rangle$. Let $\boldsymbol{w} \in V^{\perp}$. Since $\mathbb{Q}\langle\mathcal{K}\rangle = \mathbb{Q}\langle V\rangle^{\perp}$, there exist $\gamma_1, \ldots, \gamma_s \in \mathbb{Q}$ such that

$$\boldsymbol{w} = \gamma_1\boldsymbol{a}_1 + \cdots + \gamma_s\boldsymbol{a}_s\,.$$

Let a be a common denominator for $\gamma_1, \ldots, \gamma_s$. Then, $a\boldsymbol{w} \in \mathbb{Z}\langle\mathcal{K}\rangle$. This concludes the proof of Fact 4.

Since $V^{\perp}/\mathbb{Z}\langle\mathcal{K}\rangle$ is a finite set, there exist $\boldsymbol{w}_1,\ldots,\boldsymbol{w}_u$ such that

$$V^{\perp} = \bigcup_{i=1}^{u}(\boldsymbol{w}_i + \mathbb{Z}\langle\mathcal{K}\rangle)\,.$$

For every i, $1 \le i \le u$, we define the set

$$\mathcal{Z}_{0,i} := \{l \in \mathcal{Z}_0 \,:\, \boldsymbol{k}_l \in \boldsymbol{w}_i + \mathbb{Z}\langle\mathcal{K}\rangle\}\ .$$

Since for every $l \in \mathbb{N}$, we have $\boldsymbol{k}_l \in V^{\perp}$, it follows that $\mathcal{Z}_0 = \bigcup_{i=1}^{u}\mathcal{Z}_{0,i}$. Since $\mathcal{Z}_0$ is piecewise syndetic, it follows from Property (ii) of Lemma 6.2 that there exists some index i_0 such that $\mathcal{Z}_{0,i_0}$ is piecewise syndetic. We then define

$$\mathcal{Z}_1 := \mathcal{Z}_{0,i_0}\,.$$

Fact 5. Let $l', l \in \mathcal{Z}_1$ with $l' \ge l$, and define $\boldsymbol{k} = \boldsymbol{k}_{l'} - \boldsymbol{k}_l$. Then we have $\sigma_{\boldsymbol{k}}(\mathcal{I}_{\mathcal{Z}_0}) \subset \mathcal{I}_{\mathcal{Z}_0}$.

Since $l' \ge l$, we have $\boldsymbol{k} \in V_{+}^{\perp}$, which ensures that the map $\sigma_{\boldsymbol{k}}$ is well-defined. Let $P \in \mathcal{I}_{\mathcal{Z}_0}$. The definition of $\mathcal{Z}_1$ implies that $\boldsymbol{k} = \boldsymbol{k}_{l'} - \boldsymbol{k}_l \in \mathbb{Z}\langle\mathcal{K}\rangle$. Thus, we can decompose $\boldsymbol{k}$ as

$$\boldsymbol{k} = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_u - \boldsymbol{a}_{u+1} - \cdots - \boldsymbol{a}_v\,,$$

where$\boldsymbol{a}_1,\ldots,\boldsymbol{a}_v \in \mathcal{K}$. By recursively applying Fact 2 with $\sigma_{\boldsymbol{a}_1},\ldots,\sigma_{\boldsymbol{a}_u}$, we deduce from (8.12) that $\sigma_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_u}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. On the other hand, we have $\sigma_{\boldsymbol{a}_{u+1}+\cdots+\boldsymbol{a}_v}(\sigma_{\boldsymbol{k}}(P)) = \sigma_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_u}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. By recursively applying Fact 3 with $\sigma_{\boldsymbol{a}_{u+1}},\ldots,\sigma_{\boldsymbol{a}_v}$ and using (8.12), we conclude that $\sigma_{\boldsymbol{k}}(P) \in \mathcal{I}_{\mathcal{Z}_0}$. This completes the proof of Fact 5.

We are now ready to conclude the proof of Lemma 8.10. Let $P \in \mathcal{I}$, $\boldsymbol{\phi} \in \mathrm{GL}_t(\mathbb{K})$ be a relation matrix, and $\boldsymbol{k} = \boldsymbol{k}_{l'} - \boldsymbol{k}_l$ with $l, l' \in \mathcal{Z}_1$ and $l' \ge l$. Let $b(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ be a nonzero polynomial such that $bP \in \overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{Y}]$. By Lemma 8.4, we have $bP \in \mathcal{I}_{\mathcal{Z}_0}$. Then, by Fact 5, we deduce that $\sigma_{\boldsymbol{k}}(bP) \in \mathcal{I}_{\mathcal{Z}_0} \subset \mathcal{I}$. By Lemma 8.8, we have

$$b(T_{\boldsymbol{k}}\boldsymbol{z})P(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) = \sigma_{\boldsymbol{k}}(bP)(\boldsymbol{\phi},\boldsymbol{z}) = 0\,.$$

Since $T_{\boldsymbol{k}}$ is nonsingular and $b(\boldsymbol{z}) \ne 0$, it follows that $P(\boldsymbol{\phi}\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}}\boldsymbol{z}) = 0$, which completes the proof. □

Until the end of the proof of Theorem 7.2, we fix a piecewise syndetic set $\mathcal{Z}_1 \subset \mathcal{Z}_0$ that satisfies Lemma 8.10.

8.4. **Analyticity of relation matrices.** Let $\boldsymbol{\phi}$ be a relation matrix. All coordinates of $\boldsymbol{\phi}$ being algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$, they generate a finite extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let $\mathbf{k} \subset \mathbb{K}$ denote this extension and let $\gamma \ge 1$ be the degree of $\mathbf{k}$. Choosing a primitive element π in $\mathbf{k}$, we obtain a decomposition of the form

$$\boldsymbol{\phi} = \sum_{j=0}^{\gamma-1}\boldsymbol{\phi}_j(\boldsymbol{z})\pi^j\,, \tag{8.14}$$

where the matrices $\boldsymbol{\phi}_j(\boldsymbol{z})$, $0 \le j \le \gamma - 1$, have coefficients in $\overline{\mathbb{Q}}(\boldsymbol{z})$. The field $\mathbb{K}$ is *a priori* an abstract algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$, but we can easily reduce the situation to the case where the coordinates of $\boldsymbol{\phi}$ are analytic at some suitable point $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$.

Lemma 8.11. *We continue with the previous notation. There exist an integer $l_0 \in \mathcal{Z}_1$, a neighborhood $\mathcal{V}$ of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, and a function $\varphi(\boldsymbol{z})$ that is analytic on $\mathcal{V}$ and algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$ such that the following properties holds.*

(a) $\|T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}\| < 1$.
(b) $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$ *belongs to the disc of convergence of* $f_{1,1}(\boldsymbol{z}), \ldots, f_{r,m_r}(\boldsymbol{z})$.
(c) *The matrix*

$$\boldsymbol{\phi}(\boldsymbol{z}) := \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(\boldsymbol{z})\varphi(\boldsymbol{z})^j \in \mathrm{GL}_t(\mathcal{M}er(\mathcal{V}))$$

is a relation matrix. That is, it satisfies Lemmas 8.8 and 8.10.
(d) *For every j, $0 \leq j \leq \gamma - 1$, the coordinates of the matrix $\boldsymbol{\phi}_j(\boldsymbol{z})$ are analytic on $\mathcal{V}$ and the matrix $\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})$ is invertible.*

Proof. By Theorem 5.10, $\lim_{l\to\infty} T_{\boldsymbol{k}_l}\boldsymbol{\alpha} = 0$. This ensures that (a) and (b) are satisfied when l_0 is large enough.

Let $P(\boldsymbol{z}, y) \in \overline{\mathbb{Q}}[\boldsymbol{z}, y]$ denote the minimal polynomial of π and let $D(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ denote the discriminant of P, seen as a polynomial in the variable y. Since $\boldsymbol{\phi}$ is a relation matrix, $\det \boldsymbol{\phi}$ is nonzero and algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$. There thus exist polynomials $q_0(\boldsymbol{z}), \ldots, q_\nu(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$, $q_0(\boldsymbol{z}) \neq 0$, such that

$$q_0(\boldsymbol{z}) = q_1(\boldsymbol{z}) \det \boldsymbol{\phi} + q_2(\boldsymbol{z}) \det \boldsymbol{\phi}^2 + \cdots + q_\nu(\boldsymbol{z}) \det \boldsymbol{\phi}^\nu \,. \tag{8.15}$$

Let $d(\boldsymbol{z})$ be the least common multiple of the denominators of the coefficients of the matrices $\boldsymbol{\phi}_0(\boldsymbol{z}), \ldots, \boldsymbol{\phi}_{\gamma-1}(\boldsymbol{z})$. Since the polynomial $D(\boldsymbol{z})q_0(\boldsymbol{z})d(\boldsymbol{z})$ is nonzero, Theorem 6.4 ensures the existence of a full set $\mathcal{E} \subset \mathbb{N}$ such that

$$D(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})q_0(T_{\boldsymbol{k}_l}\boldsymbol{\alpha})d(T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0 \,, \; \forall l \in \mathcal{E} \,.$$

Since $\mathcal{E}$ is full and $\mathcal{Z}_1$ is piecewise syndetic, the set $\mathcal{E} \cap \mathcal{Z}_1$ is piecewise syndetic (cf. Remark 8.1). Let $l_0 \in \mathcal{E} \cap \mathcal{Z}_1$ be large enough to guarantee that (a) and (b) hold. Since $D(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, the implicit function theorem (see, for instance, [21, Proposition 6.1, p. 138]) implies that there exists a function $\varphi(\boldsymbol{z})$ that is analytic on a neighborhood of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, say $\mathcal{V}_0$, and such that $P(\boldsymbol{z}, \varphi(\boldsymbol{z})) = 0$. Note that there is a $\overline{\mathbb{Q}}(\boldsymbol{z})$-isomorphism between the field $\overline{\mathbb{Q}}(\boldsymbol{z}, \pi)$ and $\overline{\mathbb{Q}}(\boldsymbol{z}, \varphi(\boldsymbol{z}))$. We thus deduce that the matrix $\boldsymbol{\phi}(\boldsymbol{z}) := \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(\boldsymbol{z})\varphi(\boldsymbol{z})^j$ satisfies the properties of Lemmas 8.8 and 8.10. Furthermore, as $\det \boldsymbol{\phi} \neq 0$, we also deduce that $\det \boldsymbol{\phi}(\boldsymbol{z}) \neq 0$. Hence $\boldsymbol{\phi}(\boldsymbol{z}) \in \mathrm{GL}_t(\mathcal{M}er(\mathcal{V}_0))$. Finally, we deduce that $\det \boldsymbol{\phi}(\boldsymbol{z})$ also satisfies Equation (8.15). Since $q_0(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, we get that $\det \boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$. Hence the matrix $\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})$ is invertible. Furthermore, since $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$, the coordinates of $\boldsymbol{\phi}_j(\boldsymbol{z})$ are analytic on some neighborhood of $T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}$, say $\mathcal{V}_1$. Finally, setting $\mathcal{V} := \mathcal{V}_0 \cap \mathcal{V}_1$, we obtain that Properties (a)–(d) are satisfied. □

9. Proof of Theorem 7.2

We continue with the notation of Sections 7 and 8. We recall that $P_\star \in \overline{\mathbb{Q}}[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r]$, defined in Section 7.2, is a polynomial of total degree d_i in the indeterminates $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,m_i})$ satisfying

$$P_\star(f_{1,1}(\boldsymbol{\alpha}_1), \ldots, f_{r,m_r}(\boldsymbol{\alpha}_r)) = 0 \,.$$

We recall that $M = m_1 + \cdots + m_r$ is the number of functions $f_{i,j}$, which also corresponds to the number of indeterminates in $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$, while $N = n_1 + \cdots + n_r$ is the number of indeterminates $z_{i,j}$. The monomials $\boldsymbol{X}^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{X}^{\boldsymbol{\mu}_t}$, $\boldsymbol{\mu}_j \in \mathbb{N}^M$, $1 \leq j \leq t$, are precisely those which are of total degree d_i in the indeterminates $\boldsymbol{X}_i$, for every $i \in \{1, \ldots, r\}$. We also recall that, according to the notation of Section 4.1.4, if

$$\boldsymbol{\mu} = (\mu_{1,1}, \ldots, \mu_{1,m_1}, \mu_{2,1}, \ldots, \mu_{r,m_r}) \in \mathbb{N}^M ,$$

then

$$\boldsymbol{X}^{\boldsymbol{\mu}} = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq m_i}} X_{i,j}^{\mu_{i,j}} \quad \text{and} \quad \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}} = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq m_i}} f_{i,j}(\boldsymbol{z}_i)^{\mu_{i,j}} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\} .$$

Hence there exist $\tau_1, \ldots, \tau_t \in \overline{\mathbb{Q}}$ such that

$$P_\star(\boldsymbol{X}) = \sum_{j=1}^{t} \tau_j \boldsymbol{X}^{\boldsymbol{\mu}_j} .$$

Given a matrix of indeterminates $\boldsymbol{Y} := (y_{i,j})_{1 \leq i,j \leq t}$, we set

$$F(\boldsymbol{Y}, \boldsymbol{z}) := \sum_{1 \leq i,j \leq t} \tau_i y_{i,j} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_j} \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}] . \tag{9.1}$$

Note that F is a linear form in $\boldsymbol{Y}$. Evaluating at $(I_t, \boldsymbol{\alpha})$, we obtain

$$F(I_t, \boldsymbol{\alpha}) = \sum_{j=1}^{t} \tau_j \boldsymbol{f}(\boldsymbol{\alpha})^{\boldsymbol{\mu}_j} = P_\star(f_{1,1}(\boldsymbol{\alpha}_1), \ldots, f_{r,m_r}(\boldsymbol{\alpha}_r)) = 0 . \tag{9.2}$$

Remark 9.1. We have $F(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{f}(\boldsymbol{z})] \subset \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]$. Also, $F(\boldsymbol{Y}, \boldsymbol{z})$ can be seen as an element of $\overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$, as we will sometimes do in what follows.

9.1. **Iterated relations.** Using Equalities (7.6) and (7.7), we obtain

$$\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_j} = (A_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{f}(T_{\boldsymbol{k}} \boldsymbol{z}))^{\boldsymbol{\mu}_j} = \sum_{l=1}^{t} R_{j,l}(A_{\boldsymbol{k}}(\boldsymbol{z})) \boldsymbol{f}(T_{\boldsymbol{k}} \boldsymbol{z})^{\boldsymbol{\mu}_l} , \tag{9.3}$$

for every j, $1 \leq j \leq t$. We deduce from (9.3) that

$$\boldsymbol{g}(\boldsymbol{z}) = \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{g}(T_{\boldsymbol{k}} \boldsymbol{z}) , \tag{9.4}$$

for all $\boldsymbol{k} \in \mathbb{N}^r$, where $\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ is defined as in (7.10) and

$$\boldsymbol{g}(\boldsymbol{z}) := {}^{\mathrm{t}}(\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\mu}_t}) .$$

For every i, $1 \leq i \leq r$, let $b_i(\boldsymbol{z}_i) \in \overline{\mathbb{Q}}[\boldsymbol{z}_i]$ denote the least common multiple of the denominators of the coordinates of $A_i(\boldsymbol{z}_i)$. Hence the matrix $b_i(\boldsymbol{z}_i) A(\boldsymbol{z}_i)$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}_i]$. For every $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we set

$$b_{\boldsymbol{k}}(\boldsymbol{z}) := \prod_{i=1}^{r} \prod_{j=0}^{k_i - 1} b_i(T_i^j \boldsymbol{z}_i)^{d_i} ,$$

so that the matrix $b_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z})$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}]$. For future reference, we note that for every $\boldsymbol{k}$ and $\boldsymbol{k}' \in \mathbb{N}^r$:

$$b_{\boldsymbol{k}+\boldsymbol{k}'}(\boldsymbol{z}) = b_{\boldsymbol{k}}(\boldsymbol{z}) b_{\boldsymbol{k}'}(T_{\boldsymbol{k}} \boldsymbol{z}) . \tag{9.5}$$

For all $\boldsymbol{k} \in \mathbb{N}^r$, setting $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_t) \in \overline{\mathbb{Q}}^t$, Equality (9.4) implies the following equality in $\overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]$:

$$\begin{aligned} F(\boldsymbol{Y} b_{\boldsymbol{k}}(\boldsymbol{z}), \boldsymbol{z}) &= \boldsymbol{\tau} \boldsymbol{Y} b_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{g}(\boldsymbol{z}) \\ &= \boldsymbol{\tau} \boldsymbol{Y} b_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{g}(T_{\boldsymbol{k}} \boldsymbol{z}) \\ &= F(\boldsymbol{Y} b_{\boldsymbol{k}}(\boldsymbol{z}) \boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}), T_{\boldsymbol{k}} \boldsymbol{z}) \,. \end{aligned} \tag{9.6}$$

Every point $\boldsymbol{\alpha}_i$ being regular with respect to the system (7.1.i), the number $b_{\boldsymbol{k}}(\boldsymbol{\alpha})$ is nonzero for all $\boldsymbol{k} \in \mathbb{N}^r$. From (9.2) and the fact that F is linear in $\boldsymbol{Y}$, we deduce that

$$F(\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{\alpha}), T_{\boldsymbol{k}} \boldsymbol{\alpha}) = 0 \,, \ \forall \boldsymbol{k} \in \mathbb{N}^r \,. \tag{9.7}$$

9.2. **The matrices $\boldsymbol{\Theta}_l(\boldsymbol{z})$.** From now on, we fix a nonnegative integer $l_0 \in \mathcal{Z}_1$ and a relation matrix $\boldsymbol{\phi}(\boldsymbol{z})$ satisfying the properties of Lemma 8.11. Set

$$\boldsymbol{\xi} := T_{\boldsymbol{k}_{l_0}} \boldsymbol{\alpha} \,.$$

Properties (a) and (b) in Lemma 8.11 ensure the existence of a real number r_1 such that $0 < \|\boldsymbol{\xi}\| < r_1 < 1$ and such that all the power series $f_{1,1}(\boldsymbol{z}), \ldots, f_{r,m_r}(\boldsymbol{z})$ have a radius of convergence larger than r_1. Then, by Properties (c) and (d) in Lemma 8.11, we can choose a real number r_2 satisfying $0 < \|\boldsymbol{\xi}\| + r_2 < r_1$ and such that the coefficients of the matrix $\boldsymbol{\phi}(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. For every $l \geq l_0$, we set

$$\boldsymbol{\Theta}_l(\boldsymbol{z}) := \boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha}) \boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}} \boldsymbol{\alpha})^{-1} \boldsymbol{\phi}(\boldsymbol{z}) \boldsymbol{R}_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z}) \,. \tag{9.8}$$

By (7.11), we have $\boldsymbol{\Theta}_l(\boldsymbol{\xi}) = \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$, for all $l \geq l_0$.

Remark 9.2. By Lemma 8.11, the coefficients of $\boldsymbol{\Theta}_{l_0}(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. On the other hand, one has

$$\boldsymbol{\Theta}_{l+1}(\boldsymbol{z}) = \boldsymbol{\Theta}_l(\boldsymbol{z}) \boldsymbol{R}_{\boldsymbol{k}_{l+1} - \boldsymbol{k}_l}(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}) \,, \ \forall l \geq l_0 \,. \tag{9.9}$$

This implies that, for every $l \geq l_0$, the coefficients of $\boldsymbol{\Theta}_l(\boldsymbol{z})$ are analytic on some neighborhood of $\boldsymbol{\xi}$, that is on some polydisc $\mathcal{D}(\boldsymbol{\xi}, \eta_l) \subset \mathcal{D}(\boldsymbol{\xi}, r_2)$. Also, the coefficients of $b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z}) \boldsymbol{\Theta}_l(\boldsymbol{z})$ are analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$. In what follows, we will consider expressions of the form $F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z})$ for $l \geq l_0$. Formally, these are polynomials in $f_{1,1}(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}), \ldots, f_{r,m_r}(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z})$ and the coordinates of $\boldsymbol{\Theta}_l(\boldsymbol{z})$. Note that Property (a) in Lemma 8.11 and Lemma 6.8 imply that $\|T_{\boldsymbol{k}_l} \boldsymbol{\alpha}\| \leq \|T_{\boldsymbol{k}_{l_0}} \boldsymbol{\alpha}\|$ for $l \geq l_0$. It follows that $F(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z})$ is analytic on $\mathcal{D}(\boldsymbol{\xi}, \eta_l) \subset \mathcal{D}(\boldsymbol{\xi}, r_2)$. In addition, we have that $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z})$ is analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$. Indeed, $f_{1,1}(\boldsymbol{z}), \ldots, f_{r,m_r}(\boldsymbol{z})$ are analytic on $\mathcal{D}(\boldsymbol{0}, r_1) \supset \mathcal{D}(\boldsymbol{\xi}, r_2)$, while our choice of l_0 ensures that the coordinates of $\boldsymbol{\Theta}_{l_0}(\boldsymbol{z})$ are analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$.

9.3. **The key lemma.** In this section, we prove the following result from which we will deduce easily Theorem 7.2 in Section 9.4. Indeed, the identity in Lemma 9.3 provides an algebraic relation between the functions $f_{i,j}(\boldsymbol{z}_i)$, $1 \leq i \leq r$, $1 \leq j \leq m_i$, over an algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ and with the same shape that $P_\star$.

Lemma 9.3. *One has* $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z}) = 0$.

Let us first briefly describe the general strategy used for proving this key lemma. The scheme of the proof is classical and takes its source in the early work of Mahler [41]. It was gradually improved and refined by Kubota [33], Loxton and van der Poorten [38], and Ku. Nishioka [50, 51, 52]. It takes here a more complicate shape, involving the matrices $\mathbf{\Theta}_l$. This is due to the fact that we have to deal with a bunch of Mahler systems of the form (7.1.i) without any restriction on the matrices $A_i(\boldsymbol{z}_i)$.

Assuming by contradiction that $F(\mathbf{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z}) \neq 0$, we construct, for every triple of nonnegative integers (δ_1,δ_2,l), $l \geq l_0$, an auxiliary function of the form

$$E(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{j=0}^{\delta_1} P_j(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})F(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})^j\,,$$

where $P_j(\boldsymbol{Y},\boldsymbol{z})$ is a polynomial of degree at most δ_1 in each indeterminate $y_{i,j}$ and of total degree at most δ_2 in the indeterminates $z_{i,j}$. Recall that $E(\mathbf{\Theta}_l(\boldsymbol{\xi}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi}) = E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ and that

$$F(\mathbf{\Theta}_l(\boldsymbol{\xi}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi}) = F(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,.$$

Hence $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = P_0(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$. As our construction ensures that $P_0 \notin \mathcal{I}$, we have $P_0(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0$ for infinitely many integers $l \in \mathcal{Z}_1$, and we can use Liouville's inequality (4.2) to find a lower bound for $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$. On the other hand, we have many choices for the polynomials P_i in the construction of our auxiliary function. This level of freedom is used to show that, for a good choice of such polynomials, the quantity $E(R_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})$ is small enough. More precisely, we obtain a contradiction between the upper and the lower bounds when the parameter δ_1 is sufficiently large, the parameter δ_2 is sufficiently large with respect to δ_1, and the parameter $l \in \mathcal{Z}_1$ is sufficiently large with respect to δ_1 and δ_2.

Warning. The auxiliary function $E(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ can be thought of as a simultaneous Padé approximant of type I for the first δ_1 powers of $F(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$. However, we have to be careful: $F(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ is not necessarily a power series in $\boldsymbol{z}$. It is a linear combination of the power series $f_{1,1}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}),\ldots,f_{r,m_r}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ whose coefficients are only known to be algebraic over $\overline{\mathbb{Q}}(\boldsymbol{z})$. We only know that it is analytic at $\boldsymbol{\xi}$. In order to obtain our upper bound, we will have to consider the Taylor expansion of $E(\mathbf{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ (and other related functions) at ξ. This is a new feature of our proof, since until now, Mahler's method only used Taylor expansions at 0.

In what follows, we argue by contradiction, assuming that

$$F(\mathbf{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z}) \neq 0\,. \tag{9.10}$$

We divide the proof of Lemma 9.3 into four steps.

9.3.1. *First step: construction of the auxiliary function.* Given a formal power series $E = \sum_{\boldsymbol{\lambda}} e_{\boldsymbol{\lambda}}(\boldsymbol{Y})\boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$ and an integer $p > 0$, we let

$$E_p := \sum_{|\boldsymbol{\lambda}|<p} e_{\boldsymbol{\lambda}}(\boldsymbol{Y})\boldsymbol{z}^{\boldsymbol{\lambda}} \in \overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]$$

denote the truncation of E at order p with respect to $\boldsymbol{z}$. We recall that the ideal $\mathcal{I}$ is defined in Section 8.1, while the vector spaces $\mathcal{I}(\delta_1,\delta_2)$ and $\mathcal{I}^c(\delta_1,\delta_2)$ are defined in Section 8.2. In particular, $\mathcal{I}^c(\delta_1,\delta_2)$ is a vector space complement to $\mathcal{I}(\delta_1,\delta_2)$ in $\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}$. We also recall that $\mathcal{Z}_1\subset\mathcal{Z}_0\subset\mathbb{N}$ is a piecewise syndetic set that satisfies Lemma 8.10.

Lemma 9.4. *Let $\delta_1\geq 0$ be an integer. For all integers δ_2, $\delta_2\gg\delta_1$, there exist polynomials $P_i\in\mathcal{I}^c(\delta_1,\delta_2)$, $0\leq i\leq\delta_1$, not all zero, such that the formal power series*

$$E'(\boldsymbol{Y},\boldsymbol{z}):=\sum_{j=0}^{\delta_1}P_j(\boldsymbol{Y},\boldsymbol{z})F(\boldsymbol{Y},\boldsymbol{z})^j\in\overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]]$$

satisfies $E'_p(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0$, $\forall l\in\mathcal{Z}_1$, $l\geq l_0$, where $p=\left\lfloor\frac{\delta_1^{1/N}\delta_2}{2^{(t^2+2)/N}}\right\rfloor$.

Proof. Set

$$\mathcal{J}(\delta_1,\delta_2):=\{P\in\overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{Y}]\ :\ P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y},\boldsymbol{z})\in\mathcal{I}(\delta_1,\delta_2)\}\,.$$

The $\overline{\mathbb{Q}}$-vector spaces $\mathcal{J}(\delta_1,\delta_2)$ and $\mathcal{I}(\delta_1,\delta_2)$ have the same dimension. This follows directly from the fact that the map

$$\begin{array}{ccc}\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2} & \to & \overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{\delta_1,\delta_2}\\ P(\boldsymbol{Y},\boldsymbol{z}) & \mapsto & P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y},\boldsymbol{z})\end{array}$$

is an isomorphism, since the matrix $\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}$ is invertible. The latter property follows from Lemmas 7.3 and 8.11. Furthermore, we have

$$P(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0\,,\ \forall P\in\mathcal{J}(\delta_1,\delta_2),\ \forall l\in\mathcal{Z}_1,\ l\geq l_0\,. \tag{9.11}$$

Indeed, if $P\in\mathcal{J}(\delta_1,\delta_2)$, then $P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{Y},\boldsymbol{z})\in\mathcal{I}(\delta_1,\delta_2)$, and by applying Lemma 8.10 with $\boldsymbol{k}=\boldsymbol{k}_l-\boldsymbol{k}_{l_0}$ implies that

$$P(\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})\boldsymbol{\phi}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^{-1}\boldsymbol{\phi}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0\,.$$

By (9.8), this leads to $P(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0$, which proves (9.11).

Let p be as in the lemma and let us consider the three $\overline{\mathbb{Q}}$-linear maps:

$$\left\{\begin{array}{c}\prod_{j=0}^{\delta_1}\mathcal{I}^c(\delta_1,\delta_2)\\ (P_0(\boldsymbol{Y},\boldsymbol{z}),\ldots,P_{\delta_1}(\boldsymbol{Y},\boldsymbol{z}))\end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{c}\overline{\mathbb{Q}}[\boldsymbol{Y}]_{2\delta_1}[[\boldsymbol{z}]]\\ E'(\boldsymbol{Y},\boldsymbol{z}):=\sum_{j=0}^{\delta_1}P_j(\boldsymbol{Y},\boldsymbol{z})F(\boldsymbol{Y},\boldsymbol{z})^j\end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{c}\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,p-1}\\ E'_p(\boldsymbol{Y},\boldsymbol{z})\end{array}\right.$$

$$\downarrow$$

$$\left\{\begin{array}{c}\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,p-1}/\mathcal{J}(2\delta_1,p-1)\\ E'_p(\boldsymbol{Y},\boldsymbol{z})\quad\mathrm{mod}\ \mathcal{J}(2\delta_1,p-1)\end{array}\right.$$

Note that these maps are well-defined. By Lemma 8.6, the dimension of the $\overline{\mathbb{Q}}$-vector space $\mathcal{I}^c(\delta_1,\delta_2)$ is at least equal to $\frac{c_1(\delta_1)}{2}\delta_2^N$, assuming that δ_2 is large enough. It follows that

$$\dim_{\overline{\mathbb{Q}}}\left(\prod_{j=0}^{\delta_1}\mathcal{I}^c(\delta_1,\delta_2)\right) \geq \frac{c_1(\delta_1)}{2}(\delta_1+1)\delta_2^N\,. \tag{9.12}$$

For every pair of nonnegative integers (n,m), set

$$\overline{\mathcal{J}}(n,m) := \overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{n,m}/\mathcal{J}(n,m)\,.$$

Since $\mathcal{J}(\delta_1,\delta_2)$ and $\mathcal{I}(\delta_1,\delta_2)$ have same dimension, Lemma 8.7 implies that

$$\dim_{\overline{\mathbb{Q}}}\overline{\mathcal{J}}(2\delta_1,p-1) \leq 2^{t^2}\dim_{\overline{\mathbb{Q}}}\overline{\mathcal{J}}(\delta_1,p-1)\,.$$

Now, if δ_2 is sufficiently large, Lemma 8.6 ensures that

$$\dim_{\overline{\mathbb{Q}}}\overline{\mathcal{J}}(\delta_1,p-1) \leq 2c_1(\delta_1)p^N\,.$$

On the other hand, the choice of p ensures that

$$2^{t^2}\left(2c_1(\delta_1)p^N\right) < \frac{c_1(\delta_1)}{2}(\delta_1+1)\delta_2^N$$

and (9.12) implies that

$$\dim_{\overline{\mathbb{Q}}}\left(\prod_{j=0}^{\delta_1}\mathcal{I}^c(\delta_1,\delta_2)\right) > \dim_{\overline{\mathbb{Q}}}\left(\overline{\mathbb{Q}}[\boldsymbol{Y},\boldsymbol{z}]_{2\delta_1,p-1}/\mathcal{J}(2\delta_1,p-1)\right)\,.$$

Hence the $\overline{\mathbb{Q}}$-linear map defined by

$$(P_0(\boldsymbol{Y},\boldsymbol{z}),\ldots,P_{\delta_1}(\boldsymbol{Y},\boldsymbol{z})) \mapsto E'_p(\boldsymbol{Y},\boldsymbol{z}) \mod \mathcal{J}(2\delta_1,p-1)$$

has a nontrivial kernel. We deduce the existence of polynomials $P_0,\ldots,P_{\delta_1}$ in $\mathcal{I}^c(\delta_1,\delta_2)$, not all zero, such that $E'_p\in\mathcal{J}(2\delta_1,p-1)$. By (9.11), we obtain that $E'_p(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})=0$ for all $l\in\mathcal{Z}_1$, $l\geq l_0$. This ends the proof. □

Let E' be a formal power series satisfying the properties of Lemma 9.4 and let v_0 be the smallest index such that the polynomial P_{v_0} is nonzero:

$$v_0 := \min\{j\ :\ 0\leq j\leq\delta_1 \text{ and } P_j\neq 0\} \tag{9.13}$$

Then the formal power series

$$E(\boldsymbol{Y},\boldsymbol{z}) := \sum_{j=v_0}^{\delta_1}P_j(\boldsymbol{Y},\boldsymbol{z})F(\boldsymbol{Y},\boldsymbol{z})^{j-v_0}\in\overline{\mathbb{Q}}[\boldsymbol{Y}][[\boldsymbol{z}]] \tag{9.14}$$

is the auxiliary function that we were looking for. Note that we have

$$E(\boldsymbol{Y},\boldsymbol{z})F^{v_0}(\boldsymbol{Y},\boldsymbol{z}) = E'(\boldsymbol{Y},\boldsymbol{z})\,. \tag{9.15}$$

9.3.2. *Second step: upper bound for* $|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})|$. The aim of this section is to prove the following proposition.

Proposition 9.5. *There exists a real number* $c_2 > 0$ *that does not depend on* δ_1*,* δ_2*, and* l*, such that*

$$|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \leq e^{-c_2 e^l \delta_1^{1/N}\delta_2}\,, \quad \forall l \in \mathcal{Z}_1,\, l \gg \delta_2 \gg \delta_1\,.$$

This result will be deduced from three auxiliary results: Lemmas 9.6, 9.7, and 9.8. We first set

$$G(\boldsymbol{Y}, \boldsymbol{z}) := E'(\boldsymbol{Y}, \boldsymbol{z}) - E'_p(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]\,, \tag{9.16}$$

where p is defined as in Lemma 9.4. For every monomial $\boldsymbol{Y}^{\boldsymbol{\nu}}$ appearing in $G(\boldsymbol{Y}, \boldsymbol{z})$, the $t \times t$ matrix $\boldsymbol{\nu}$ has a decomposition

$$\boldsymbol{\nu} = \Xi + \Pi$$

where Ξ has coefficients in the set $\{0, 1, \ldots, \delta_1\}$ and Π is a matrix with nonnegative integer coefficients whose sum is at most δ_1. In the definition of E', the polynomials P_j contribute to Ξ, while the powers of F contribute to Π. Let s denote the number of such matrices $\boldsymbol{\nu}$ and $\boldsymbol{Y}^{\boldsymbol{\nu}_1}, \ldots, \boldsymbol{Y}^{\boldsymbol{\nu}_s}$ denote an enumeration of the corresponding monomials. The sum of the coefficients of each matrix $\boldsymbol{\nu}_i$ is at most equal to $(t^2+1)\delta_1$. By (9.16), there exists a unique decomposition of the form

$$G(\boldsymbol{Y}, \boldsymbol{z}) = \sum_{i=1}^{s} \sum_{|\boldsymbol{\lambda}| \geq p} g_{\boldsymbol{\lambda},i} \boldsymbol{z}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\nu}_i}\,,$$

where $g_{\boldsymbol{\lambda},i} \in \mathbb{C}$. For every i, $1 \leq i \leq s$, we define the formal power series

$$G_i(\boldsymbol{z}) := \sum_{|\boldsymbol{\lambda}| \geq p} g_{\boldsymbol{\lambda},i} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \mathbb{C}[[\boldsymbol{z}]]\,.$$

We recall that the positive real numbers r_1 and r_2 are defined in Section 9.2. By definition of $F(\boldsymbol{Y}, \boldsymbol{z})$, these series belong to $\overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{f}(\boldsymbol{z})]$. In particular, they are analytic on some polydisc $\mathcal{D}(\boldsymbol{0}, r)$ with $r > r_1$. It follows that every function $G_i(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}\boldsymbol{z})$, $1 \leq i \leq s$, $l \geq l_0$, is analytic on the polydisc $\mathcal{D}(\boldsymbol{\xi}, r_2)$, and hence has a Taylor expansion at $\boldsymbol{\xi}$ which is absolutely convergent on $\mathcal{D}(\boldsymbol{\xi}, r_2)$:

$$G_i(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} h_{\boldsymbol{\lambda},i,l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{9.17}$$

where $h_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$.

Lemma 9.6. *There exists a real number* $\sigma > 0$ *such that*

$$|h_{\boldsymbol{\lambda},i,l}| \leq e^{-\sigma e^l p}\,,\ \forall l \gg \delta_1, \delta_2, \boldsymbol{\lambda}\,.$$

Proof. Since $G_i(\boldsymbol{z})$ is analytic on some polydisc $\mathcal{D}(\boldsymbol{0}, r)$ with $r > r_1$, we infer from (4.1) that there exists a real number $\sigma_1(\delta_1, \delta_2) > 0$ such that

$$|g_{\boldsymbol{\lambda},i}| \leq \sigma_1(\delta_1, \delta_2) r_1^{-|\boldsymbol{\lambda}|}\,,\ \forall \boldsymbol{\lambda} \in \mathbb{N}^N\,. \tag{9.18}$$

On the other hand, we claim that there exists a real number $\sigma_2 > 0$ such that

$$|\boldsymbol{\lambda} T_{\boldsymbol{k}_l}| \geq \sigma_2 e^l |\boldsymbol{\lambda}| \quad \forall l \in \mathbb{N}, \forall \boldsymbol{\lambda} \in \mathbb{N}^N. \tag{9.19}$$

Let us prove this claim. Write $\boldsymbol{k}_l = (k_{1,l}, \ldots, k_{r,l})$ and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_r)$, with $\boldsymbol{\lambda}_i$ a row vector of size n_i. By Lemma 5.6, for each i, $1 \leq i \leq r$, there exists a real number $\sigma_{3,i} > 0$, independent of $\boldsymbol{\lambda}_i$, such that

$$|\boldsymbol{\lambda}_i T_i^{k_{i,l}}| \geq \sigma_{3,i} \rho(T_i)^{k_{i,l}} |\boldsymbol{\lambda}_i| , \quad \forall l \in \mathbb{N} .$$

Then we infer from Property (ii) of Definition 7.4, that there exists a real number $\sigma_4 > 0$ such that $\rho(T_i)^{k_{i,l}} \geq \sigma_4 e^l$. Since

$$|\boldsymbol{\lambda}| = \sum_{i=1}^{r} |\boldsymbol{\lambda}_i| \quad \text{and} \quad |\boldsymbol{\lambda} T_{\boldsymbol{k}_l}| = \sum_{i=1}^{r} |\boldsymbol{\lambda}_i T_i^{k_{i,l}}| ,$$

the lower bound in (9.19) holds with $\sigma_2 := \sigma_4 \min(\sigma_{3,i} : 1 \leq i \leq r)$.

Thus, for every $l \geq l_0$, $G_i(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z})$ can be written as

$$G_i(T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}) = \sum_{|\boldsymbol{\lambda}| \geq \sigma_2 e^l p} g_{\boldsymbol{\lambda},i,l} \boldsymbol{z}^{\boldsymbol{\lambda}} , \tag{9.20}$$

with $g_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$. Furthermore, this power series is absolutely convergent on the polydisc $\mathcal{D}(\mathbf{0}, r_1)$. Since the matrix $T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}$ is invertible and has nonnegative integer coefficients, $|\boldsymbol{\mu}| \leq |\boldsymbol{\mu} T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}|$ for all $\boldsymbol{\mu}$. When $\boldsymbol{\lambda} = \boldsymbol{\mu} T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}$ for some $\boldsymbol{\mu}$ and some $l \geq l_0$, we deduce from (9.18) and the fact that $r_1 < 1$ that

$$|g_{\boldsymbol{\lambda},i,l}| = |g_{\boldsymbol{\mu},i}| \leq \sigma_1(\delta_1, \delta_2) r_1^{-|\boldsymbol{\mu}|} \leq \sigma_1(\delta_1, \delta_2) r_1^{-|\boldsymbol{\mu} T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}|} = \sigma_1(\delta_1, \delta_2) r_1^{-|\boldsymbol{\lambda}|} ,$$

for all $i \in \{1, \ldots, s\}$. Since $g_{\boldsymbol{\lambda},i,l} = 0$ when $\boldsymbol{\lambda}$ is not of the previous form, we have

$$|g_{\boldsymbol{\lambda},i,l}| \leq \sigma_1(\delta_1, \delta_2) r_1^{-|\boldsymbol{\lambda}|} \tag{9.21}$$

for all $\boldsymbol{\lambda} \in \mathbb{N}^N$, $i \in \{1, \ldots, s\}$, and $l \geq l_0$.

Since by assumption $\mathcal{D}(\boldsymbol{\xi}, r_2) \subset \mathcal{D}(\mathbf{0}, r_1)$, the two power series expansions (9.17) and (9.20) match on $\mathcal{D}(\boldsymbol{\xi}, r_2)$. Using the equality $\boldsymbol{z}^{\boldsymbol{\lambda}} = ((\boldsymbol{z} - \boldsymbol{\xi}) + \boldsymbol{\xi})^{\boldsymbol{\lambda}}$ and identifying, for every $\boldsymbol{\lambda} \in \mathbb{N}^N$, the terms in $(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}$ in (9.17) and (9.20), we obtain that

$$h_{\boldsymbol{\lambda},i,l} = \sum_{\substack{|\boldsymbol{\gamma}| \geq \sigma_2 e^l p \\ \boldsymbol{\gamma} \geq \boldsymbol{\lambda}}} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} g_{\boldsymbol{\gamma},i,l} \boldsymbol{\xi}^{\boldsymbol{\gamma} - \boldsymbol{\lambda}} . \tag{9.22}$$

We observe that for $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N) \geq \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$, one has

$$\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} = \prod_{i=1}^{N} \frac{\gamma_i!}{(\gamma_i - \lambda_i)! \lambda_i!} \leq \prod_{i=1}^{N} \gamma_i^{\lambda_i} \leq |\boldsymbol{\gamma}|^{|\boldsymbol{\lambda}|} . \tag{9.23}$$

Let $\boldsymbol{\lambda} \in \mathbb{N}^N$. If l is large enough, then $|\boldsymbol{\lambda}| < \sigma_2 e^l p$ and we infer from (9.21), (9.22), and (9.23) that

$$\begin{aligned} |h_{\boldsymbol{\lambda},i,l}| &\leq \sum_{|\boldsymbol{\gamma}|\geq \sigma_2 e^l p} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} |g_{\boldsymbol{\gamma},i,l}| \, \|\boldsymbol{\xi}\|^{|\boldsymbol{\gamma}-\boldsymbol{\lambda}|} \\ &\leq \sum_{|\boldsymbol{\gamma}|\geq \sigma_2 e^l p} |\boldsymbol{\gamma}|^{|\boldsymbol{\lambda}|} \sigma_1(\delta_1,\delta_2) \left(\frac{\|\boldsymbol{\xi}\|}{r_1}\right)^{|\boldsymbol{\gamma}|} \|\boldsymbol{\xi}\|^{-|\boldsymbol{\lambda}|} \\ &\leq \sigma_1(\delta_1,\delta_2)\|\boldsymbol{\xi}\|^{-|\boldsymbol{\lambda}|} \left(\sum_{|\boldsymbol{\gamma}|\geq \sigma_2 e^l p} \left(|\boldsymbol{\gamma}|^{|\boldsymbol{\lambda}|} \left(\frac{\|\boldsymbol{\xi}\|}{r_1}\right)^{|\boldsymbol{\gamma}|/2} \right) \left(\frac{\|\boldsymbol{\xi}\|}{r_1}\right)^{|\boldsymbol{\gamma}|/2} \right) . \end{aligned}$$

Since $\|\boldsymbol{\xi}\| < r_1$, we have $(\|\boldsymbol{\xi}\|/r_1)^{1/2} < 1$. Thus, if l is large enough with respect to $|\boldsymbol{\lambda}|$, there exists a real number $\sigma_5 > 0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l such that

$$|h_{\boldsymbol{\lambda},i,l}| \leq \sigma_1(\delta_1,\delta_2)\|\boldsymbol{\xi}\|^{-|\boldsymbol{\lambda}|} \left(\sigma_5 \left(\frac{\|\boldsymbol{\xi}\|}{r_1}\right)^{(\sigma_2 e^l p)/2} \right) .$$

Finally, we obtain that there exists a real number $\sigma_6 > 0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l such that

$$|h_{\boldsymbol{\lambda},i,l}| \leq e^{-\sigma_6 e^l p} , \ \forall l \gg \delta_1, \delta_2, \boldsymbol{\lambda} .$$

Setting $\sigma := \sigma_6$, this ends the proof. □

By Remark 9.2, the monomial $(b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z}))^{\boldsymbol{\nu}_i}$ is analytic on $\mathcal{D}(\boldsymbol{\xi}, r_2)$ for every i, $1 \leq i \leq s$, and every $l \geq l_0$. Multiplying by $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1-|\boldsymbol{\nu}_i|}$, we obtain on $\mathcal{D}(\boldsymbol{\xi}, r_2)$ a Taylor expansion of the form

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1-|\boldsymbol{\nu}_i|}(b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z}))^{\boldsymbol{\nu}_i} = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} \theta_{\boldsymbol{\lambda},i,l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}} , \tag{9.24}$$

where $\theta_{\boldsymbol{\lambda},i,l} \in \mathbb{C}$.

Lemma 9.7. *There exists a real number $\kappa(\delta_1, \boldsymbol{\lambda}) > 0$ that depends on δ_1 and $\boldsymbol{\lambda}$ but not on l, such that*

$$|\theta_{\boldsymbol{\lambda},i,l}| \leq e^{\kappa(\delta_1,\boldsymbol{\lambda})l} , \ \ \forall i,\ 1 \leq i \leq s,\ \forall \boldsymbol{\lambda} \in \mathbb{N}^N,\ \forall l \geq l_0 .$$

Proof. Given $Q(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ with total degree d_Q and $l \geq 0$, the polynomial $Q_l(\boldsymbol{z}) := Q(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$ can be converted into a polynomial in $(\boldsymbol{z}-\boldsymbol{\xi})$:

$$Q_l(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}} q_{\boldsymbol{\lambda},l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}$$

and we claim that there exists a real number $\kappa_1(\boldsymbol{\lambda}) > 0$ that depends on $\boldsymbol{\lambda}$ (and Q) but not on l such that

$$|q_{\boldsymbol{\lambda},l}| \leq \kappa_1(\boldsymbol{\lambda})e^{|\boldsymbol{\lambda}|l} . \tag{9.25}$$

Let us prove this claim. By Lemma 6.5, there exists a real number $\kappa_2 > 0$ such that

$$|\boldsymbol{\lambda}T_{\boldsymbol{k}_l}| \leq \kappa_2 e^l |\boldsymbol{\lambda}| \quad \forall l \in \mathbb{N}, \forall \boldsymbol{\lambda} \in \mathbb{N}^N. \tag{9.26}$$

Let κ_3 denote the maximum of the modulus of the coefficients of Q. Then, using (9.23), (9.26), and the equality $\boldsymbol{z}^{\boldsymbol{\lambda}} = ((\boldsymbol{z}-\boldsymbol{\xi})+\boldsymbol{\xi})^{\boldsymbol{\lambda}}$, we obtain that

$$\begin{aligned}|q_{\boldsymbol{\lambda},l}| &\leq \kappa_3 \sum_{\substack{\boldsymbol{\gamma}\,:\,|\boldsymbol{\gamma}|\leq d_Q \text{ and} \\ \boldsymbol{\gamma}T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\geq \boldsymbol{\lambda}}} \binom{\boldsymbol{\gamma}T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}}{\boldsymbol{\lambda}} \|\boldsymbol{\xi}\|^{|\boldsymbol{\gamma}T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}|-|\boldsymbol{\lambda}|} \\ &\leq \kappa_3 \sum_{\boldsymbol{\gamma}\,:\,|\boldsymbol{\gamma}|\leq d_Q} |\boldsymbol{\gamma}T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}|^{|\boldsymbol{\lambda}|} \\ &\leq \kappa_3 \binom{d_Q+N+1}{N+1}(\kappa_2 e^l d_Q)^{|\boldsymbol{\lambda}|}\,.\end{aligned}$$

Setting $\kappa_1(\boldsymbol{\lambda}) := \kappa_3\binom{d_Q+N+1}{N+1}(\kappa_2 d_Q)^{|\boldsymbol{\lambda}|}$, this proves (9.25).

Now, set $\Gamma_{\boldsymbol{k}}(\boldsymbol{z}) := b_{\boldsymbol{k}}(\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(\boldsymbol{z}) \in \mathcal{M}_t(\overline{\mathbb{Q}}[\boldsymbol{z}])$ and

$$\Gamma_{\boldsymbol{k},l}(\boldsymbol{z}) := \Gamma_{\boldsymbol{k}}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = b_{\boldsymbol{k}}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})\,.$$

By Properties (ii) of Definition 7.4, the sequence $(\boldsymbol{k}_{l+1}-\boldsymbol{k}_l)_{l\in\mathbb{N}}$ takes its values in a finite set. We thus deduce from our claim that, for every $\boldsymbol{\lambda}\in\mathbb{N}^N$, there exists a positive real number $\kappa_4(\delta_1,\boldsymbol{\lambda})>0$, that depends on δ_1 and $\boldsymbol{\lambda}$ but not on l, such that the modulus of the coefficients in $(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}$ for the entries of the matrices

$$b_{\boldsymbol{k}_{l+1}-\boldsymbol{k}_l}(T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})^{(t^2+1)\delta_1-|\boldsymbol{\nu}_i|}\Gamma_{\boldsymbol{k}_{l+1}-\boldsymbol{k}_l,l}(\boldsymbol{z})^{\boldsymbol{\nu}_i}, \quad 1\leq i\leq s\,,$$

seen as polynomials in $\boldsymbol{z}-\boldsymbol{\xi}$, are at most $\kappa_4(\delta_1,\boldsymbol{\lambda})e^{|\boldsymbol{\lambda}|l}$ for any $l\geq l_0$. Without any loss of generality, we can assume that $\kappa_4(\delta_1,\boldsymbol{\lambda})\geq 1$. On the other hand, we deduce from (9.5) and (9.9) the recurrence relation

$$b_{\boldsymbol{k}_{l+1}-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_{l+1}(\boldsymbol{z}) = b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})\boldsymbol{\Theta}_l(\boldsymbol{z})\Gamma_{\boldsymbol{k}_{l+1}-\boldsymbol{k}_l,l}(\boldsymbol{z})\,. \tag{9.27}$$

Now, for every positive integer δ_1, we introduce the real numbers $\kappa_5(\delta_1,\boldsymbol{\lambda})>0$ that depend on δ_1 and $\boldsymbol{\lambda}$ but not on l, which are recursively defined by

$$e^{\kappa_5(\delta_1,\boldsymbol{\lambda})} = \max\left\{\max_{1\leq i\leq s}|\theta_{\boldsymbol{\lambda},i,l_0}|^{\frac{1}{l_0}};\, 2\kappa_4(\delta_1,\boldsymbol{0});\, 2\sum_{\boldsymbol{\gamma}<\boldsymbol{\lambda}}\kappa_4(\delta_1,\boldsymbol{\lambda}-\boldsymbol{\gamma})e^{\kappa_5(\delta_1,\boldsymbol{\gamma})+|\boldsymbol{\lambda}-\boldsymbol{\gamma}|}\right\}$$

for every $\boldsymbol{\lambda}\in\mathbb{N}^N$. We prove by induction on $l\geq l_0$ that

$$|\theta_{\boldsymbol{\lambda},i,l}| \leq e^{\kappa_5(\delta_1,\boldsymbol{\lambda})l}\,,\quad \forall i,\ 1\leq i\leq s,\ \forall\boldsymbol{\lambda}\in\mathbb{N}^N,\ \forall l\geq l_0\,. \tag{9.28}$$

By definition of $\kappa_5(\delta_1,\boldsymbol{\lambda})$, Inequality (9.28) holds when $l=l_0$. Suppose that it holds for some $l\geq l_0$. Then, we infer from (9.24) and (9.27) that

$$\begin{aligned}|\theta_{\boldsymbol{\lambda},i,l+1}| &\leq \sum_{\boldsymbol{\gamma}\leq\boldsymbol{\lambda}}|\theta_{\boldsymbol{\gamma},i,l}|\kappa_4(\delta_1,\boldsymbol{\lambda}-\boldsymbol{\gamma})e^{|\boldsymbol{\lambda}-\boldsymbol{\gamma}|l} \\ &\leq \kappa_4(\delta_1,\boldsymbol{0})e^{\kappa_5(\delta_1,\boldsymbol{\lambda})l} + \left(\sum_{\boldsymbol{\gamma}<\boldsymbol{\lambda}}\kappa_4(\delta_1,\boldsymbol{\lambda}-\boldsymbol{\gamma})e^{\kappa_5(\delta_1,\boldsymbol{\gamma})+|\boldsymbol{\lambda}-\boldsymbol{\gamma}|}\right)^l \\ &\leq \frac{1}{2}e^{\kappa_5(\delta_1,\boldsymbol{\lambda})(l+1)} + \frac{1}{2}e^{\kappa_5(\delta_1,\boldsymbol{\lambda})l} \leq e^{\kappa_5(\delta_1,\boldsymbol{\lambda})(l+1)}\,.\end{aligned}$$

Hence Inequality (9.28) holds for all $l \geq l_0$. Setting $\kappa(\delta_1, \boldsymbol{\lambda}) := \kappa_5(\delta_1, \boldsymbol{\lambda})$, this ends the proof. $\square$

Now, let us observe that

$$b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1} E'(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z})$$

defines an analytic function on $\mathcal{D}(\boldsymbol{\xi}, r_2)$. Then, on this polydisc, it has a Taylor expansion

$$b_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1} E'(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \epsilon_{\boldsymbol{\lambda}, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}, \tag{9.29}$$

where $\epsilon_{\boldsymbol{\lambda}, l} \in \mathbb{C}$.

Lemma 9.8. *Let p be defined as in Lemma 9.4. There exists a real number $\gamma > 0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l, such that*

$$|\epsilon_{\boldsymbol{\lambda}, l}| \leq e^{-\gamma e^l p}, \ \forall l \in \mathcal{Z}_1, l \gg \delta_1, \delta_2, \boldsymbol{\lambda}.$$

Proof. By (9.16), we have

$$G(\boldsymbol{Y}, \boldsymbol{z}) = E'(\boldsymbol{Y}, \boldsymbol{z}) - E'_p(\boldsymbol{Y}, \boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}[\boldsymbol{Y}]$$

and hence Lemma 9.4 implies that

$$G(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}) = E'(\boldsymbol{\Theta}_l(\boldsymbol{z}), T_{\boldsymbol{k}_l - \boldsymbol{k}_{l_0}} \boldsymbol{z}), \ \forall l \in \mathcal{Z}_1, \ l \geq l_0. \tag{9.30}$$

From (9.17), (9.24), (9.29), and (9.30), we deduce that

$$\sum_{i=1}^{s} \left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} h_{\boldsymbol{\lambda}, i, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}} \right) \left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \theta_{\boldsymbol{\lambda}, i, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}} \right) = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \epsilon_{\boldsymbol{\lambda}, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}. \tag{9.31}$$

Moreover, since the power series involved in (9.17) and (9.24) are absolutely convergent on $\mathcal{D}(\boldsymbol{\xi}, r_2)$, we obtain that

$$\begin{aligned} &\sum_{i=1}^{s} \left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} h_{\boldsymbol{\lambda}, i, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}} \right) \left(\sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \theta_{\boldsymbol{\lambda}, i, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}} \right) \\ &\qquad = \sum_{\boldsymbol{\lambda} \in \mathbb{N}^N} \sum_{i=1}^{s} \sum_{\boldsymbol{\gamma} \leq \boldsymbol{\lambda}} h_{\boldsymbol{\gamma}, i, l} \theta_{\boldsymbol{\lambda} - \boldsymbol{\gamma}, i, l} (\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}. \end{aligned} \tag{9.32}$$

Finally, identifying the terms in $(\boldsymbol{z} - \boldsymbol{\xi})^{\boldsymbol{\lambda}}$ in (9.31) thanks to (9.32), we have

$$\epsilon_{\boldsymbol{\lambda}, l} = \sum_{i=1}^{s} \sum_{\boldsymbol{\gamma} \leq \boldsymbol{\lambda}} h_{\boldsymbol{\gamma}, i, l} \theta_{\boldsymbol{\lambda} - \boldsymbol{\gamma}, i, l}.$$

Now, we fix $\boldsymbol{\lambda} \in \mathbb{N}^N$. We infer from Lemmas 9.6 and 9.7 that there exists a real number $\gamma > 0$ that does not depend on δ_1, δ_2, $\boldsymbol{\lambda}$, and l, such that

$$|\epsilon_{\boldsymbol{\lambda}, l}| \leq e^{-\gamma e^l p}, \quad \forall l \in \mathcal{Z}_1, \ l \gg \delta_1, \delta_2, \boldsymbol{\lambda},$$

which ends the proof. $\square$

Proof of Proposition 9.5. Let us observe that the function

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})$$

is analytic on $\mathcal{D}(\boldsymbol{\xi},\boldsymbol{r}_2)$, so it has a Taylor expansion on this polydisc of the form

$$b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} e_{\boldsymbol{\lambda},l}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{9.33}$$

with $e_{\boldsymbol{\lambda},l}\in\mathbb{C}$. Let v_0 be defined by (9.13). Then $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0}$ is analytic on $\mathcal{D}(\boldsymbol{\xi},r_2)$, so that

$$F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0} = \sum_{\boldsymbol{\lambda}\in\mathbb{N}^N} a_{\boldsymbol{\lambda}}(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}}\,, \tag{9.34}$$

with $a_{\boldsymbol{\lambda}}\in\mathbb{C}$. Using (9.6), we get that

$$F(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z}) = F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})\,,\quad \forall l\geq l_0\,.$$

This equality is *a priori* valid for $\boldsymbol{z}$ in $\mathcal{D}(\boldsymbol{\xi},\eta_l)$ (see Remark 9.2), but it extends to $\mathcal{D}(\boldsymbol{\xi},r_2)$ by analytic continuation. By (9.15), we thus have

$$E(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})^{v_0} = E'(\boldsymbol{\Theta}_l(\boldsymbol{z}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{z})\,, \tag{9.35}$$

for all $l\geq l_0$ and all $\boldsymbol{z}\in\mathcal{D}(\boldsymbol{\xi},\eta_l)$. By (9.10), $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}),\boldsymbol{z})$ is nonzero and there thus exists at least one nonzero coefficient $a_{\boldsymbol{\lambda}}$ in (9.34). Let us consider an index $\boldsymbol{\lambda}_0$ such that $a_{\boldsymbol{\lambda}_0}\neq 0$ with $|\boldsymbol{\lambda}_0|$ minimal. Multiplying both sides of (9.35) by $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}$, and identifying the coefficients in $(\boldsymbol{z}-\boldsymbol{\xi})^{\boldsymbol{\lambda}_0}$ in their power series expansion on $\mathcal{D}(\boldsymbol{\xi},r_2)$ with the help of (9.29), (9.33), and (9.34), we obtain that

$$e_{0,l}a_{\boldsymbol{\lambda}_0} = \epsilon_{\boldsymbol{\lambda}_0,l}\,,\quad \forall l\geq l_0\,.$$

Since $\boldsymbol{\Theta}_l(\boldsymbol{\xi}) = \boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha})$, we infer from Lemma 9.8 and the fact that $p = \left\lfloor\frac{\delta_1^{1/N}\delta_2}{2^{(t^2+2)/N}}\right\rfloor$ (cf. Lemma 9.4), the existence of a real number $\beta_1>0$ that does not depend on δ_1, δ_2, and l, such that

$$\begin{aligned}
&\left|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}),T_{\boldsymbol{k}_l}\boldsymbol{\alpha})\right|\\
&= \left|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}E(\boldsymbol{\Theta}_l(\boldsymbol{\xi}),T_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}\boldsymbol{\xi})\right|\\
&= |e_{0,l}|\\
&\leq e^{-\beta_1 e^l\delta_1^{1/N}\delta_2}\,,\quad \forall l\in\mathcal{Z}_1,\, l\gg\delta_1,\delta_2\,.
\end{aligned} \tag{9.36}$$

Now, it just remains to find a lower bound for $|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}|$. By (9.26), there exists a positive real number β_2 that does not depend on l, δ_1, and δ_2, such that the degree of the polynomial $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{(t^2+1)\delta_1}$ is at most equal to $\beta_2 e^l\delta_1$. Furthermore, the height of its coefficients it at most equal to $\beta_3^l\delta_1$ for some positive real number β_3. Thus, there exists a positive real number β_4 such that

$$\log H(b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})^{(t^2+1)\delta_1}) \leq \beta_4 e^l\delta_1\,. \tag{9.37}$$

Each $\boldsymbol{\alpha}_i$ being by assumption regular w.r.t. (7.1.i), $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})\neq 0$ for all $l\geq l_0$. Since the numbers $b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})$, $l\geq l_0$, belong to some fixed number field, we infer from (9.37) and Liouville's inequality (4.2) the existence of a real number $\beta_5 > 0$ that does not depend on l, δ_1, and δ_2, such that

$$|b_{\boldsymbol{k}_l-\boldsymbol{k}_{l_0}}(\boldsymbol{\xi})|^{(t^2+1)\delta_1} \geq e^{-\beta_5 e^l \delta_1}\,, \quad \forall l \geq l_0,\, \forall \delta_1 \in \mathbb{N}\,. \tag{9.38}$$

By (9.36) and (9.38), there exists a real number $\beta_6 > 0$ that does not depend on l, δ_1, and δ_2, such that

$$\begin{aligned} |E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| &\leq e^{\beta_5 e^l \delta_1 - \beta_1 e^l \delta_1^{1/N}\delta_2} \\ &\leq e^{-\beta_6 e^l \delta_1^{1/N}\delta_2}\,, \quad \forall l \in \mathcal{Z}_1, l \gg \delta_2 \gg \delta_1\,. \end{aligned}$$

Setting $c_2 := \beta_6$, this completes the proof. □

9.3.3. *Third step: lower bound for* $|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})|$. The aim of this section is to prove the following proposition.

Proposition 9.9. *There exists a real number $c_3 > 0$ that does not depend on δ_1, δ_2, l, and an infinite set of positive integers $\mathcal{E} \subset \mathcal{Z}_1$ such that*

$$|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \geq e^{-c_3 e^l \delta_2}\,, \quad \forall l \in \mathcal{E}, \delta_2 \geq \delta_1\,.$$

Proof. Let us first recall that, by (9.7), we have

$$F(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\,, \quad \forall l \in \mathbb{N}\,.$$

By construction of our auxiliary function (cf. (9.14)), we deduce that

$$E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})\,,$$

where v_0 is defined as in (9.13). Furthermore, Lemma 9.4 ensures that $P_{v_0} \notin \mathcal{I}$ and $P_{v_0} \in \overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{Y}]$. By Property (iii) of Lemma 8.4 , we have that the set

$$\mathcal{Z}(P_{v_0}) = \{l \in \mathcal{Z}_0 : P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) = 0\}$$

is negligible. By Remark 8.1, the set $\mathcal{E} := \mathcal{Z}_1 \setminus \mathcal{Z}(P_{v_0})$ is piecewise syndetic since $\mathcal{Z}_1$ is piecewise syndetic. In particular, $\mathcal{E}$ is an infinite set. Since $P_{v_0} \in \overline{\mathbb{Q}}[\boldsymbol{z}, \boldsymbol{Y}]$ and $\mathcal{Z}_1 \subset \mathcal{Z}_0$, it follows that

$$P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha}) \neq 0\,, \quad \forall l \in \mathcal{E}\,.$$

Since P_{v_0} has degree at most δ_1 in each indeterminate $y_{i,j}$, and total degree at most δ_2 in $\boldsymbol{z}$, Liouville's inequality (4.2) and a computation similar to the previous one ensure the existence of a real number $c_3 > 0$ that does not depend on δ_1, δ_2, and l, such that

$$|E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| = |P_{v_0}(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \geq e^{-c_3 e^l \delta_2}\,, \quad \forall l \in \mathcal{E}, \delta_2 \geq \delta_1\,,$$

as wanted. □

9.3.4. *Fourth step: contradiction.* By Propositions 9.5 and 9.9, we obtain that

$$e^{-c_3 e^l \delta_2} \leq |E(\boldsymbol{R}_{\boldsymbol{k}_l}(\boldsymbol{\alpha}), T_{\boldsymbol{k}_l}\boldsymbol{\alpha})| \leq e^{-c_2 e^l \delta_1^{1/N}\delta_2}\,, \quad \forall l \in \mathcal{E},\, l \gg \delta_2 \gg \delta_1\,.$$

Finally, we deduce that

$$c_3 \geq c_2 \delta_1^{1/N}\,.$$

Since c_2 and c_3 do not depend on δ_1, this inequality provides a contradiction, as soon as δ_1 is large enough. This completes the proof of Lemma 9.3. □

9.4. **End of the proof of Theorem 7.2.** Let us recall that $d(\boldsymbol{z}) \in \overline{\mathbb{Q}}[\boldsymbol{z}]$ stands for the least common multiple of the denominators of the coefficients of the matrices $\boldsymbol{\phi}_j(\boldsymbol{z})$ defined in (8.14). Hence Property (c) of Lemma 8.11 implies that $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$ has coefficients in $\overline{\mathbb{Q}}[\boldsymbol{z}, \varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})]$, while Property (d) of Lemma 8.11 ensures that $d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) \neq 0$. Let $q(\boldsymbol{z})$ denote the least common multiple of the denominators of the coefficients of the matrix $\boldsymbol{R}_{\boldsymbol{k}_{l_0}}^{-1}(\boldsymbol{z})$. By Lemma 7.3, we have $q(\boldsymbol{\alpha}) \neq 0$.

By Lemma 9.3, we know that $F(\boldsymbol{\Theta}_{l_0}(\boldsymbol{z}), \boldsymbol{z}) = 0$, and substituting $T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}$ to $\boldsymbol{z}$, we obtain that $F(\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}), T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}) = 0$. The function $F(\boldsymbol{Y}, \boldsymbol{z})$ being linear in $\boldsymbol{Y}$, we deduce that

$$F\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}), T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}\right) = 0\,.$$

Writing $\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}) = \boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1}\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})$, and using (9.6), we get that

$$F\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1}, \boldsymbol{z}\right) = 0\,. \tag{9.39}$$

Set

$$Q_\star(\boldsymbol{z}, \boldsymbol{X}) := \boldsymbol{\tau}\left(\frac{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})q(\boldsymbol{z})}{b_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})d(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})q(\boldsymbol{\alpha})}\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})\boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{z})^{-1}\right)\mathbf{V}\,,$$

where

$$\mathbf{V} := {}^{\mathrm{t}}(\boldsymbol{X}^{\boldsymbol{\mu}_1}, \ldots, \boldsymbol{X}^{\boldsymbol{\mu}_t})\,.$$

It follows that $Q_\star(\boldsymbol{z}, \boldsymbol{X}) \in \overline{\mathbb{Q}}[\boldsymbol{z}, \varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z}), \boldsymbol{X}]$. Since $\varphi \circ T_{\boldsymbol{k}_{l_0}}$ is analytic at $\boldsymbol{\alpha}$, we deduce that $Q_\star(\boldsymbol{z}, \boldsymbol{X}) \in \overline{Q(\boldsymbol{z})}_{\boldsymbol{\alpha}}[\boldsymbol{X}]$. Moreover, since $\boldsymbol{\Theta}_{l_0}(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha}) = \boldsymbol{\Theta}_{l_0}(\boldsymbol{\xi}) = \boldsymbol{R}_{\boldsymbol{k}_{l_0}}(\boldsymbol{\alpha})$, we obtain that

$$Q_\star(\boldsymbol{\alpha}, \boldsymbol{X}) = \boldsymbol{\tau}\mathbf{V} = \sum_{j=1}^{t}\tau_j\boldsymbol{X}^{\boldsymbol{\mu}_j} = P_\star(\boldsymbol{X})\,.$$

Finally, it follows from (9.39) that

$$Q_\star(\boldsymbol{z}, \boldsymbol{f}(\boldsymbol{z})) = 0\,.$$

This ends the first part of the proof of Theorem 7.2.

Now, let us assume that $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$ is a regular extension of $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let $\mathbb{K}$ be an algebraic closure of $\overline{\mathbb{Q}}(\boldsymbol{z})$ containing $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$. By [34, Chapter VIII], $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$ and $\mathbb{K}$ are linearly disjoint over $\overline{\mathbb{Q}}(\boldsymbol{z})$. Let δ denote the degree of $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})$ over $\overline{\mathbb{Q}}(\boldsymbol{z})$. Since the functions $\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})^j$, $0 \leq j \leq \delta - 1$, are linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})$, they remain linearly independent over $\overline{\mathbb{Q}}(\boldsymbol{z})(\boldsymbol{f}(\boldsymbol{z}))$. Splitting the polynomial $Q_\star$ as

$$Q_\star = \sum_{j=0}^{\delta-1} Q_j(\boldsymbol{z}, \boldsymbol{X})\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{z})^j\,,$$

where $Q_j(\boldsymbol{z},\boldsymbol{X}) \in \overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{X}]$, we thus deduce that $Q_j(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z}))=0$ for all j, $0 \le j \le \delta-1$. Finally, setting

$$R_\star(\boldsymbol{z},\boldsymbol{X}) := \sum_{j=0}^{\delta-1} Q_j(\boldsymbol{z},\boldsymbol{X})\varphi(T_{\boldsymbol{k}_{l_0}}\boldsymbol{\alpha})^j \in \overline{\mathbb{Q}}[\boldsymbol{z},\boldsymbol{X}]\,,$$

we obtain that $R_\star(\boldsymbol{z},\boldsymbol{f}(\boldsymbol{z}))=0$ and $R_\star(\boldsymbol{\alpha},\boldsymbol{X})=P_\star(\boldsymbol{X})$, as wanted. □

10. Proofs of Theorems 3.3, 3.6, 3.8, and of Corollaries 3.5 and 3.9

In this section, we complete the proof of our main results. Note that we establish Corollary 3.9 before Theorems 3.6 and 3.8. The latter are in fact deduced from Corollary 3.9.

10.1. **Proof of Theorem 3.3.** There is nothing more to do. Theorem 3.3 simply corresponds to the case $r=1$ of Theorem 7.2. □

10.2. **Proof of Corollary 3.5.** We first note that the inequality

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})) \le \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))$$

always holds. Recall that $\boldsymbol{z}=(z_1,\ldots,z_n)$. When $n=1$, this inequality is trivial. The general case can be proved by induction on n, arguing as in the proof of Lemma 10.1. Let $\mathbb{K}$ denote the field of fractions of the ring $\overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}$. Since $\mathbb{K}$ is algebraic over $\overline{\mathbb{Q}}(z)$, we have

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z})) = \mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))\,.$$

It thus remains to prove that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}),\ldots,f_m(\boldsymbol{\alpha})) \ge \mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}),\ldots,f_m(\boldsymbol{z}))\,. \tag{10.1}$$

We follow the same strategy as the one used for proving the Siegel–Shidlovskii theorem in the framework of E-functions (see [27, Theorem 5.23, p. 230]). We first replace Proposition 5.1 in [27] by the following result.

Lemma 10.1. *Let us continue with the previous notation. Let us assume that $g_1(\boldsymbol{z}),\ldots,g_\ell(\boldsymbol{z}) \in \overline{\mathbb{Q}}\{\boldsymbol{z}\}$ are related by a linear T-Mahler system, that $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is regular w.r.t. this system, and that the pair $(T,\boldsymbol{\alpha})$ is admissible. Let s be the maximum number of functions among $g_1(\boldsymbol{z}),\ldots,g_\ell(\boldsymbol{z})$ that are linearly independent over $\mathbb{K}$. Then at least s of the numbers $g_1(\boldsymbol{\alpha}),\ldots,g_\ell(\boldsymbol{\alpha})$ are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. The integer s is the dimension of the $\mathbb{K}$-vector space spanned by $g_1(\boldsymbol{z}),\ldots,g_\ell(\boldsymbol{z})$. Let t denote the dimension of the $\overline{\mathbb{Q}}$-vector space spanned by the numbers $g_i(\boldsymbol{\alpha})$, $1 \le i \le \ell$. Hence we have to prove that $t \ge s$. Since the dimension of the dual vector spaces are respectively equal to $\ell-s$ and $\ell-t$, it is enough to find $\ell-t$ linearly independent forms in the dual of the $\mathbb{K}$-vector space spanned by $g_1(\boldsymbol{z}),\ldots,g_\ell(\boldsymbol{z})$.

Without any loss of generality, we can assume that $g_1(\boldsymbol{\alpha}),\ldots,g_t(\boldsymbol{\alpha})$ are linearly independent over $\overline{\mathbb{Q}}$. Then, for every i, $t<i\le\ell$, there exist algebraic numbers $\gamma_{i,1},\ldots,\gamma_{i,t}$ such that

$$g_i(\boldsymbol{\alpha}) = \gamma_{i,1}g_1(\boldsymbol{\alpha}) + \cdots + \gamma_{i,t}g_t(\boldsymbol{\alpha})\,.$$

By Theorem 3.3, there exist $p_{i,1}(\boldsymbol{z}),\ldots,p_{i,\ell}(\boldsymbol{z})\in\overline{\mathbb{Q}(\boldsymbol{z})}_{\boldsymbol{\alpha}}\subset\mathbb{K}$ such that

$$p_{i,1}(\boldsymbol{z})g_1(\boldsymbol{z})+\cdots+p_{i,\ell}(\boldsymbol{z})g_\ell(\boldsymbol{z})=0\,,$$

with $p_{i,j}(\boldsymbol{\alpha})=-\gamma_{i,j}$ when $1\leq j\leq t$, $p_{i,i}(\boldsymbol{\alpha})=1$ and $p_{i,j}(\boldsymbol{\alpha})=0$ when $t<j\leq\ell$ and $j\neq i$. Set

$$L_i(\boldsymbol{z},X_1,\ldots,X_\ell):=p_{i,1}(\boldsymbol{z})X_1+\cdots+p_{i,\ell}(\boldsymbol{z})X_\ell\,,\ t<i\leq\ell\,.$$

Note that the linear form $L_i(\boldsymbol{\alpha},X_1,\ldots,X_m)$ is equal to

$$X_i-\sum_{j=1}^{t}\gamma_{i,j}X_j\,.$$

Hence the linear forms $L_i(\boldsymbol{\alpha},X_1,\ldots,X_m)$, $t<i\leq\ell$, are linearly independent over $\overline{\mathbb{Q}}$. We stress that it implies that the corresponding linear forms $L_i(\boldsymbol{z},X_1,\ldots,X_\ell)$ are linearly independent over $\mathbb{K}$.

We first prove that the linear forms

$$L_i(z_1,\alpha_2,\ldots,\alpha_n,X_1,\ldots,X_m),\quad t<i\leq\ell\,, \tag{10.2}$$

are linearly independent over $\mathbb{K}$. We argue by contradiction, assuming that they are linearly dependent over $\mathbb{K}$. Since these linear forms do not depend on the variables $z_2,\ldots,z_n$, it follows that they are also linearly dependent over $\mathbb{K}\cap\overline{\mathbb{Q}}\{z_1-\alpha_1\}$. Hence there exist $c_{t+1}(z_1),\ldots,c_\ell(z_1)\in\mathbb{K}\cap\overline{\mathbb{Q}}\{z_1-\alpha_1\}$ such that

$$\sum_{i=t+1}^{\ell}c_i(z_1)L_i(z_1,\alpha_2,\ldots,\alpha_n,X_1,\ldots,X_m)=0\,. \tag{10.3}$$

Note that if some $h(z_1)\in\mathbb{K}\cap\overline{\mathbb{Q}}\{z_1-\alpha_1\}$ is such that $h(\alpha_1)=0$ then $h(z_1)/(z_1-\alpha_1)\in\mathbb{K}\cap\overline{\mathbb{Q}}\{z_1-\alpha_1\}$. Thus, dividing (10.3) by a power of $(z_1-\alpha_1)$ if necessary, we can assume that not all the $c_i(z_1)$ vanish at α_1. Then, specializing (10.3) at α_1 we obtain the nontrivial relation

$$\sum_{i=t+1}^{\ell}c_i(\alpha_1)L_i(\alpha_1,\alpha_2,\ldots,\alpha_n,X_1,\ldots,X_m)=0\,,$$

which contradicts the fact that the forms $L_i(\boldsymbol{\alpha},X_1,\ldots,X_m)$ are linearly independent over $\overline{\mathbb{Q}}$. Thus, the forms defined in (10.2) are linearly independent over $\mathbb{K}$, as claimed.

Now, let us show in a similar way that the linear forms

$$L_i(z_1,z_2,\alpha_3,\ldots,\alpha_n,X_1,\ldots,X_m),\quad t<i\leq\ell\,,$$

are linearly independent over $\mathbb{K}$. We argue by contradiction, assuming that they are linearly dependent over $\mathbb{K}$. Since these linear forms do not depend on the variables $z_3,\ldots,z_n$, it follows that they are also linearly dependent over $\mathbb{K}\cap\overline{\mathbb{Q}}\{(z_1-\alpha,z_2-\alpha_2)\}$. Hence, there exist $e_{t+1}(z_1,z_2),\ldots,e_\ell(z_1,z_2)\in\mathbb{K}\cap\overline{\mathbb{Q}}\{(z_1-\alpha,z_2-\alpha_2)\}$ such that

$$\sum_{i=t+1}^{\ell}e_i(z_1,z_2)L_i(z_1,z_2,\alpha_3,\ldots,\alpha_n,X_1,\ldots,X_m)=0\,. \tag{10.4}$$

As previously, dividing (10.4) by a power of $(z_2 - \alpha_2)$ if necessary, we can assume that not all functions $e_i(z_1, \alpha_2)$ are zero, so that we derive from (10.4) the nontrivial relation

$$\sum_{i=t+1}^{\ell} e_i(z_1, \alpha_2) L_i(z_1, \alpha_2, \ldots, \alpha_n, X_1, \ldots, X_m) = 0\,,$$

which contradicts the fact, previously established, that the forms

$$L_i(z_1, \alpha_2, \ldots, \alpha_n, X_1, \ldots, X_m), \quad t < i \leq \ell\,,$$

are linearly independent over $\mathbb{K}$.

Continuing to argue in the same way, we obtain recursively that the $\ell - t$ linear forms

$$L_i(z_1, \ldots, z_n, X_1, \ldots, X_m), \quad t < i \leq \ell\,,$$

are linearly independent over $\mathbb{K}$. Thus $\ell - s \geq \ell - t$ and hence $t \geq s$, as wanted. □

Proof of Corollary 3.5. Let $D \geq 0$ be an integer. As in [27, p. 231], we let $\varphi_{\boldsymbol{\alpha}}(D)$ denote the dimension of the $\overline{\mathbb{Q}}$-vector space spanned by the monomials of total degree at most D in $f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})$. We also let $\varphi_{\boldsymbol{z}}(D)$ denote the dimension of the $\mathbb{K}$-vector space spanned by the monomials of total degree at most D in $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$. Now, we observe that the monomials of total degree at most D in $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ are also related by a linear T-Mahler system for which $\boldsymbol{\alpha}$ remains regular. Indeed, such a system can be obtained by taking the system associated with the matrix $((1) \oplus A(\boldsymbol{z}))^{\otimes D}$, that is, the Dth power of Kronecker of the matrix $(1) \oplus A(\boldsymbol{z})$, where $A(\boldsymbol{z})$ is defined as in (3.1) (see [12, p. 17]). Then, we infer from Lemma 10.1 that

$$\varphi_{\boldsymbol{\alpha}}(D) \geq \varphi_{\boldsymbol{z}}(D),\ \forall D \in \mathbb{N}\,. \tag{10.5}$$

By the Hilbert-Serre Theorem (cf. [56, Theorem 42, p. 235]), for sufficiently large D, $\varphi_{\boldsymbol{\alpha}}(D)$ and $\varphi_{\boldsymbol{z}}(D)$ are polynomials in D whose degree are respectively $\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha}))$ and $\mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))$. Hence (10.5) implies that

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_m(\boldsymbol{\alpha})) \geq \mathrm{tr.deg}_{\mathbb{K}}(f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}))\,,$$

as wanted. □

10.3. **Proof of Corollary 3.9.** We first need the two following simple results.

Lemma 10.2. *Let $\boldsymbol{z} = (z_{1,1}, \ldots, z_{1,n_1}, z_{2,1}, \ldots, z_{r,n_r})$ be a tuple of $n_1 + \cdots + n_r$ distinct variables. For every i, $1 \leq i \leq r$, let $f_{i,1}(\boldsymbol{z}_i), \ldots, f_{i,m_i}(\boldsymbol{z}_i) \in \overline{\mathbb{Q}}[[\boldsymbol{z}_i]]$ be some power series, where $\boldsymbol{z}_i = (z_{i,1}, \ldots, z_{i,n_i})$. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq i \leq r,\ 1 \leq j \leq m_i\} = \sum_{i=1}^{r} \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_i)}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq j \leq m_i\}\,.$$

Proof. The result follows directly from the fact that the sets of variables $\{z_{1,1}, \ldots, z_{1,n_1}\}, \ldots, \{z_{r,1}, \ldots, z_{r,n_r}\}$ are disjoint. □

Lemma 10.3. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_r, \mathcal{F}_1, \ldots, \mathcal{F}_r$ be nonempty finite sets of complex numbers such that $\mathcal{E}_i \subset \mathcal{F}_i$ for every i, $1 \leq i \leq r$. Let us assume that*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{F}_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_i)\,.$$

Then

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,.$$

Proof. Suppose first that all elements of each $\mathcal{F}_i$, $1 \leq i \leq r$, are algebraically independent. By assumption, all elements of the set $\bigcup_{i=1}^{r} \mathcal{F}_i$ are algebraically independent. Hence, all elements of the set $\bigcup_{i=1}^{r} \mathcal{E}_i$ are also algebraically independent, and the lemma is proved. Let us assume now that some elements of the family $\mathcal{F}_i$, $1 \leq i \leq r$, are algebraically dependent. For every i, we choose a subset of algebraically independent elements $\mathcal{E}_i' \subset \mathcal{E}_i$ such that $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i') = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$. Then, we complete the set $\mathcal{E}_i'$ in a set of algebraically independent elements $\mathcal{F}_i' \subset \mathcal{F}_i$ such that $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_i') = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_i)$. From the first part of the proof, we have

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i'\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i')\,.$$

It follows that

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i\right) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{i=1}^{r} \mathcal{E}_i'\right) = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i') = \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,,$$

which ends the proof. □

We are now ready to prove Corollary 3.9.

Proof of Corollary 3.9. We continue with the notation of Theorems 3.6 and 3.8.

Let us first assume that the assumptions of Theorem 3.6 are satisfied. We can gather all the linear Mahler systems (3.3.i) into a big Mahler system of the form (3.1), where $A(\boldsymbol{z}) = A_1(\boldsymbol{z}_1) \oplus \cdots \oplus A_r(\boldsymbol{z}_r)$, $\boldsymbol{z} = (z_{1,1}, \ldots, z_{r,m_r})$, and $T := T_1 \oplus \cdots \oplus T_r$. Then, we infer from assumptions (i) and (ii) of Theorem 3.6 and from Theorem 5.10 that the pair $(T, \boldsymbol{\alpha})$ is admissible and that the point $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$ is regular with respect to this T-Mahler system. Hence we can apply Corollary 3.5 to this larger system. We obtain that

$$\begin{aligned}(10.6)\ \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \leq i \leq r,\, 1 \leq j \leq m_i\}\\ = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq i \leq r,\, 1 \leq j \leq m_i\}\,.\end{aligned}$$

On the other hand, applying Corollary 3.5 to the system (3.3.i), for every i, $1 \leq i \leq r$, we deduce that

(10.7)

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i)\} : 1 \leq j \leq m_i\} = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_i)}\{f_{i,j}(\boldsymbol{z}_i) : 1 \leq j \leq m_i\}\,.$$

It follows from (10.6), (10.7), and Lemma 10.2 that

$$\text{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \leq i \leq r,\, 1 \leq j \leq m_i\} = \sum_{i=1}^{r} \text{tr.deg}_{\overline{\mathbb{Q}}}\{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \leq j \leq m_i\}\,. \tag{10.8}$$

For every i, set $\mathcal{F}_i := \{f_{i,j}(\boldsymbol{\alpha}_i) : 1 \leq j \leq m_i\}$. Using (10.8), we can thus apply Lemma 10.3 to deduce that $\text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)$, as wanted.

Now, let us assume that the assumptions of Theorem 3.8 are satisfied. The proof is essentially the same. The only change occurs when establishing Equality (10.6). We infer from assumptions (i) and (ii) of Theorem 3.8 that we can apply Theorem 7.2. Then, using Theorem 7.2 and arguing as in the proof of Corollary 3.5, we deduce that Equality (10.6) holds. The rest of the proof remains unchanged. □

10.4. **Proof of Theorems 3.6 and 3.8.** We continue with the notation of Theorems 3.6 and 3.8. Note that the inclusion

$$\sum_{i=1}^{r} \text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \subset \text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) \tag{10.9}$$

is trivial. Suppose that the assumptions of either Theorem 3.6 or Theorem 3.8 hold. By Corollary 3.9, we have

$$\text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,. \tag{10.10}$$

Given a prime ideal $\mathcal{I}$, we let $\text{ht}(\mathcal{I})$ denote its height, that is the maximum of the integers h such that there exist primes ideals $\mathfrak{p}_0, \ldots, \mathfrak{p}_h$ satisfying

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathcal{I}\,.$$

By [56, Chapter VII, Theorem 20], we have

$$\begin{aligned} \text{ht}\left(\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\right) &= s_i - \text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\,, \quad \forall i,\ 1 \leq i \leq r, \\ \text{ht}\left(\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})\right) &= S - \text{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E})\,, \end{aligned} \tag{10.11}$$

where $S := s_1 + \cdots + s_r$ and, for every i, $1 \leq i \leq r$, s_i is the number of coordinates of the tuple $\mathcal{E}_i$. Then we deduce from (10.10) and (10.11) that

$$\text{ht}\left(\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})\right) = \sum_{i=1}^{r} \text{ht}\left(\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i)\right)\,.$$

Set $\mathcal{I} := \sum_{i=1}^{r} \text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})$. Then the isomorphism[6]

$$\overline{\mathbb{Q}}[\boldsymbol{X}_1]/\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_1) \otimes_{\overline{\mathbb{Q}}} \cdots \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}[\boldsymbol{X}_r]/\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_r) \cong \overline{\mathbb{Q}}[\boldsymbol{X}]/\mathcal{I}$$

implies that $\mathcal{I}$ is a prime ideal. Indeed, the tensor product of integral domains, over an algebraically closed field, is an integral domain. Furthermore, this isomorphism also gives that $\text{ht}(\mathcal{I}) = \sum_{i=1}^{r} \text{ht}(\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i))$. It follows that $\text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E})$ and $\sum_{i=1}^{r} \text{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E})$ are both prime ideals with the same height. By (10.9) and [56, Chapter VII, §7 (2)] these two ideals are equal. This ends the proof. □

[6] See, for instance, [31, Chapter I, Exercise 3.15].

11. Proof of Theorem 1.1

In this section, we show how to deduce Theorem 1.1 from the two purity theorems. We first prove the following lemma.

Lemma 11.1. *Let $f(z)$ be an M_q-function and α be a nonzero algebraic number such that $f(z)$ is well-defined at α. Then there exists an M_q-function $g(z)$ such that the following properties hold.*

(a) $g(\alpha) = f(\alpha)$.

(b) *There exists a positive integer l such that $g(z)$ is the first coordinate of a vector solution to a q^l-Mahler system, say*

$$\begin{pmatrix} g_1(z) = g(z) \\ \vdots \\ g_m(z) \end{pmatrix} = B(z) \begin{pmatrix} g_1(z^{q^l}) \\ \vdots \\ g_m(z^{q^l}) \end{pmatrix} . \tag{11.1}$$

(c) *The point α is regular with respect to (11.1).*

Proof. We first note that if $f(\alpha)$ is algebraic, the lemma is trivial for we can choose $g(z) := f(\alpha)$ to be constant. We assume now that $f(\alpha)$ is transcendental. Replacing q by q^{l_0} for some sufficiently large l_0 if necessary, we can assume without any loss of generality that $f(z)$ is well-defined at α^{q^l} for all $l \geq 0$. Using the minimal q-Mahler equation for $f(z)$, we deduce that $f(z)$ is the first coordinate of some q-Mahler system, say

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^q) \\ \vdots \\ f_m(z^q) \end{pmatrix} , \tag{11.2}$$

where $f_1(z) := f(z), \ldots, f_m(z) := f(z^{q^{m-1}})$ are linearly independent over $\overline{\mathbb{Q}}(z)$ and well-defined at α. Thus, we infer from [8, Theorem 1.10] that there exists an integer l such that the number α^{q^l} is regular with respect to the iterated system

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A_l(z) \begin{pmatrix} f_1(z^{q^l}) \\ \vdots \\ f_m(z^{q^l}) \end{pmatrix} , \tag{11.3}$$

where

$$A_l(z) := A(z)A(z^q) \cdots A(z^{q^{l-1}}) .$$

Furthermore, [8, Theorem 1.10] also ensures that α is not a pole of the matrix $A_l(z)$. Let $(a_1(z), \ldots, a_m(z))$ denote the first row of $A_l(z)$. Set

$$g(z) := a_1(\alpha) f_1(z^{q^l}) + \cdots + a_m(\alpha) f_m(z^{q^l}) . \tag{11.4}$$

Note that $g(z)$ is an M_q-function for it is obtained as a linear combination over $\overline{\mathbb{Q}}$ of M_q-functions[7]. Since $f(\alpha)$ is transcendental, the vector $(a_1(\alpha), \ldots, a_m(\alpha))$ is nonzero. Then applying a suitable constant gauge transformation to the Mahler system associated with the matrix $A_l(z^{q^l})$, we can obtain a Mahler system which has a solution vector with $g(z)$ as first

[7]Indeed, if $f(z)$ is an M_q-function then so is $f(z^{q^l})$.

coordinate. Furthermore, since the point α^{q^l} is regular w.r.t. (11.3), α is a regular point w.r.t. this new system. On the other hand, we infer from (11.3) and (11.4) that $g(\alpha) = f(\alpha)$, as wanted. □

Proof of Theorem 1.1. We keep on with the notation of Theorem 1.1. We assume that none of the numbers $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ belong to $\mathbb{K}$, so that it remains to prove that $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$. By [8, Corollaire 1.8], our assumption implies that the numbers $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are all transcendental. For every i, $1 \leq i \leq r$, we let z_i denote an indeterminate.

By Lemma 11.1, with each pair (f_i, α_i), we can associate an M_{q_i}-function $g_i(z_i)$ such that

$$(11.5.i) \qquad \begin{pmatrix} g_{i,1}(z_i) = g_i(z_i) \\ \vdots \\ g_{i,m_i}(z_i) \end{pmatrix} = B_i(z_i) \begin{pmatrix} g_{i,1}\left(z_i^{q_i^{l_i}}\right) \\ \vdots \\ g_{i,m_i}\left(z_i^{q_i^{l_i}}\right) \end{pmatrix},$$

$g_i(\alpha_i) = f_i(\alpha_i)$, and α_i is regular w.r.t. (11.5.i).

Let us first prove Case (i) of Theorem 1.1. Let us divide the natural numbers $1, \ldots, r$ into s classes $\mathcal{I}_1 = \{i_{1,1}, \ldots, i_{1,\nu_1}\}, \ldots, \mathcal{I}_s = \{i_{s,1}, \ldots, i_{s,\nu_s}\}$, such that i and j belong to the same classe if and only if q_i and q_j are multiplicatively dependent. Iterating each system (11.5.i) a suitable number of times, we can assume without loss of generality that $q_i^{l_i} = q_j^{l_j} := \rho_k$ whenever i and j belong to $\mathcal{I}_k$. We set

$$\mathcal{E}_{k,i} := (g_i(\alpha_i)), \ \forall k \in \{1, \ldots, s\}, \ \forall i \in \mathcal{I}_k,$$

and then

$$\mathcal{E}_k := (\mathcal{E}_{k,i_{k,1}}, \ldots, \mathcal{E}_{k,i_{k,\nu_k}}) = (g_{i_{k,1}}(\alpha_{i_{k,1}}), \ldots, g_{i_{k,\nu_k}}(\alpha_{i_{k,\nu_k}})), \ \forall k \in \{1, \ldots, s\}.$$

Finally, we set

$$\mathcal{E} := (\mathcal{E}_1, \ldots, \mathcal{E}_s),$$

so that the coordinates of the r-tuple $\mathcal{E}$ are precisely $g_1(\alpha_1), \ldots, g_r(\alpha_r)$ (possibly in a different order).

Given $k \in \{1, \ldots, s\}$, we consider the Mahler system in the variables $z_i, i \in \mathcal{I}_k$, associated with the matrix $\oplus_{i \in \mathcal{I}_k} B_i(z_i)$ and the transformation $T_k = \rho_k I_{\nu_k}$, which belongs to the class $\mathcal{T}$ for $\rho_k \geq 2$. In this way, we have converted our r Mahler systems in one variable into s Mahler systems, each having respectively $\nu_1, \ldots, \nu_s$ variables. Furthermore, since by assumption the algebraic numbers $\alpha_1, \ldots, \alpha_r$ are multiplicatively independent, we deduce that each pair

$$(T_k, \boldsymbol{\alpha}_k := (\alpha_{i_{k,1}}, \ldots, \alpha_{i_{k,\nu_k}})), \ 1 \leq k \leq s,$$

is admissible. Finally, the point $\boldsymbol{\alpha}_k$ is regular since each α_i is regular w.r.t. (11.5.i). Since, by construction, the numbers $\rho(T_1) = \rho_1, \ldots, \rho(T_s) = \rho_s$ are pairwise multiplicatively independent, we can apply our second purity theorem, Theorem 3.8, to these s Mahler systems. We deduce that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{k=1}^{s} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k \mid \mathcal{E}) \,. \tag{11.6}$$

Now, let us fix $k \in \{1, \ldots, s\}$. Since the numbers $\alpha_i, i \in \mathcal{I}_k$, are multiplicatively independent, we can apply our first purity theorem, Theorem 3.6, to the ν_k distinct Mahler systems (11.5.i), with $i \in \mathcal{I}_k$. For every $i \in \mathcal{I}_k$, $g_i(\alpha_i) = f_i(\alpha_i)$ is transcendental, so that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_{k,i}) = \{0\}$. We thus deduce from Theorem 3.6 that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_k) = \sum_{i \in \mathcal{I}_k} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_{k,i} \mid \mathcal{E}_k) = \{0\} \,.$$

Since this holds for every k, $1 \leq k \leq s$, it follows from (11.6), that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \{0\}$. That is, $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$.

Now, let us prove Case (ii) of Theorem 1.1. As previously, we associate with each pair $(f_i(z), \alpha_i)$ a function $g_i(z)$ satisfying the conditions of Lemma 11.1. Since the natural numbers q_i are pairwise multiplicatively independent, we can apply our second purity theorem, Theorem 3.8, to the Mahler systems associated with each $g_i(z)$ in Lemma 11.1. Setting

$$\mathcal{E} := (g_1(\alpha_1), \ldots, g_r(\alpha_r))$$

and $\mathcal{E}_i := (g_i(\alpha_i))$, $1 \leq i \leq r$, we deduce that

$$\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^{r} \mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) \,.$$

Again, since by assumption $g_i(\alpha_i) = f_i(\alpha_i)$ is transcendental, we get that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i \mid \mathcal{E}) = \{0\}$ for every i, $1 \leq i \leq r$. This shows that $\mathrm{Alg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \{0\}$, and we conclude, as previously, that $f_1(\alpha_1), \ldots, f_r(\alpha_r)$ are algebraically independent over $\overline{\mathbb{Q}}$. □

11.1. **Comment on Theorem 1.1.** Let us just add a few words about the algorithm mentioned in Remark 1.2. The input of the algorithm is an M_q-function, say $f(z)$, and an algebraic point, say α, $0 < |\alpha| < 1$. More precisely, we assume that $f(z)$ is given by one equation of the form (1.3) and enough initial coefficients of its power series expansion so that it is uniquely defined by these data. The main issue is to compute explicitly a *good equation* satisfied by $f(z)$.

From the input data, one can first compute explicitly the *minimal inhomogeneous* q-Mahler equation satisfied by $f(z)$ (cf. [9, Algorithm 1.3]). It takes the form

$$a_{-1}(z) + a_0(z)f(z) + a_1(z)f(z^q) + \cdots + a_m(z)f(z^{q^m}) = 0 \tag{11.7}$$

where $a_i(z) \in \overline{\mathbb{Q}}[z]$, $-1 \leq i \leq m$, are relatively prime and m is minimal. Let ρ be a lower bound for the modulus of the nonzero roots of the polynomials $a_i(z)$, $-1 \leq i \leq m$, and ℓ_0 be a positive integer such that $|\alpha|^{q^{\ell_0}} < \rho$. Such an integer ℓ_0 can be effectively computed. Furthermore, $\alpha^{q^{\ell_0}}$ is regular with respect to the linear Mahler system associated with Equation (11.7), so that the lifting theorem implies that the numbers

$$1, f(\alpha^{q^{\ell_0}}), f(\alpha^{q^{\ell_0+1}}), \ldots, f(\alpha^{q^{\ell_0+m-1}})$$

are linearly independent over $\overline{\mathbb{Q}}$, since the minimality of (11.7) forces the functions $1, f(z), f(z^q), \ldots, f(z^{q^{m-1}})$ to be linearly independent over $\overline{\mathbb{Q}}(z)$. Iterating ℓ_0 times the linear Mahler system associated with Equation (11.7), we obtain a new equation of the form:

(11.8)

$$b_{-1}(z)+b_0(z)f(z)+b_1(z)f(z^{q^{\ell_0}})+b_2(z)f(z^{q^{\ell_0+1}})+\cdots+b_m(z)f(z^{q^{\ell_0+m-1}}) = 0\,,$$

with $b_i(z) \in \overline{\mathbb{Q}}[z]$, $-1 \leq i \leq m$, relatively prime. Equation (11.8), which can thus be explicitly computed, is the good equation we were looking for. Indeed, we easily deduce from (11.8) and the linear independence over $\overline{\mathbb{Q}}$ of

$$1, f(\alpha^{q^{\ell_0}}), f(\alpha^{q^{\ell_0+1}}), \ldots, f(\alpha^{q^{\ell_0+m-1}})\,,$$

the following trichotomy:

(i) $f(z)$ has a pole at α if and only if $b_0(\alpha) = 0\,,$
(ii) $f(\alpha)$ is algebraic if and only if $b_1(\alpha) = \cdots = b_m(\alpha) = 0\,,$
(iii) $f(\alpha)$ is transcendental otherwise.

It follows that one can effectively determined wether $f(\alpha)$ is transcendental or not. Furthermore, in the latter case, [8, Corollaire 1.8] implies that $f(\alpha)$ should belong to the number field generated over $\mathbb{Q}$ by the coefficients of $f(z)$ and the point α. In the end, this shows that, in the condition of Theorem 1.1, we can indeed determine wether or not one of the numbers $f_i(\alpha_i)$, $1 \leq i \leq r$, belongs to $\mathbb{K}$.

12. Proof of Theorems 2.2, 2.3, and 2.4

In this section, we show how to deduce Theorems 2.2, 2.3, and 2.4 from Theorem 1.1.

We first recall the following result due to Cobham [22]: if x is a real number whose expansion in the integer base $b \geq 2$ can be generated by a finite automaton, then there exists an M-function $f(z)$ with integer coefficients such that $x = f(1/b)$.

Proof of Theorem 2.2. Let us assume that x is a real number that is automatic in the two multiplicatively independent bases b_1 and b_2. The result of Cobham mentioned just above implies that there exist two M-functions $f_1(z)$ and $f_2(z)$ with integer coefficients such that $x = f_1(1/b_1) = f_2(1/b_2)$. Note that the coefficients of theses functions and the points $1/b_1$ and $1/b_2$ all belong to the field $\mathbb{Q}$. Since the numbers $f_1(1/b_1)$ and $f_2(1/b_2)$ are equal, they are obviously algebraically dependent over $\overline{\mathbb{Q}}$, and then Part (i) of Theorem 1.1 implies than one of them is rational. Hence x is rational. □

Proof of Theorem 2.3. Let $b_1, \ldots, b_r \geq 2$ be multiplicatively independent integers and let us assume that for every i, $1 \leq i \leq r$, x_i is automatic in base b_i. The result of Cobham mentioned just above implies that, for every i, $1 \leq i \leq r$, there exists an M-function $f_i(z)$ with integer coefficients such that $x_i = f_i(1/b_i)$. Since the coefficients of $f_i(z)$ and the point $1/b_i$ all belong to $\mathbb{Q}$, we infer from [8, Corollaire 1.8] that either $f_i(1/b_i)$ is rational or it is transcendental. Now, let us assume that $x_1, \ldots, x_r$ are all irrational. We thus deduce that these numbers are all transcendental. Since by assumption the numbers $1/b_1, \ldots, 1/b_r$ are multiplicatively independent, Part (i) of Theorem 1.1 implies that the numbers $x_1 = f_1(1/b_1), \ldots, x_r = f_r(1/b_r)$ are algebraically independent, as wanted. □

Proof of Theorem 2.4. Let us assume that the functions $f_1(z), \ldots, f_r(z)$ are all irrational. Then, we infer from [52, Theorem 5.1.7] that they are all transcendental over $\overline{\mathbb{Q}}(z)$. Combining Nishioka's theorem and [18, Lemma 6], we deduce that there exists $r > 0$ such that for all algebraic numbers α, with $0 < |\alpha| < r$, the numbers $f_1(\alpha), \ldots, f_r(\alpha)$ are all transcendental. Picking such α and applying Part (ii) of Theorem 1.1, we obtain that the numbers $f_1(\alpha), \ldots, f_r(\alpha)$ are algebraically independent over $\overline{\mathbb{Q}}$. Hence the functions $f_1(z), \ldots, f_r(z)$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, as wanted. □

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