MAHLER'S METHOD

by

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1. Preamble

Mahler's method, a term coined much later by van der Poorten, originated in three papers of Kurt Mahler [**36, 37, 38**] published in 1929 and 1930. As reported in [**42, 44**], Mahler was really sick and laid up in bed around 1926-27 when he started to occupy himself by playing with the function

$$\mathfrak{f}(z) = \sum_{n=0}^{\infty} z^{2^n} .$$

While trying to show the irrationality of the number $\mathfrak{f}(p/q)$ for rational numbers p/q with $0 < |p/q| < 1$, he finally finished proving the following much stronger statement.

Theorem 1.1. — *Let α be an algebraic number such that $0 < |\alpha| < 1$. Then $\mathfrak{f}(\alpha)$ is a transcendental number.*

And Mahler's method, an entirely new subject, was born. In the hands of Mahler, the method already culminated with the transcendence of various numbers such as

$$\sum_{n=0}^{\infty} \alpha^{2^n} , \; \prod_{n=0}^{\infty} (1-\alpha^{2^n}) , \; \sum_{n=0}^{\infty} \lfloor n\sqrt{5} \rfloor \alpha^n , \; \cfrac{1}{\alpha^{-2} + \cfrac{1}{\alpha^{-4} + \cfrac{1}{\alpha^{-8} + \cdots}}}$$

and with the algebraic independence of the numbers $\mathfrak{f}(\alpha), \mathfrak{f}'(\alpha), \mathfrak{f}"(\alpha), \ldots$. Here, α denotes again an algebraic number with $0 < |\alpha| < 1$. Furthermore, examples of this kind can be produced at will, as illustrated for instance in [**67**]. Not only Mahler's contribution was fundamental but also some of his ideas, described in [**40**], were very influential for the future development of the theory by other mathematicians. There are several surveys including a discussion on this topic, as well as seminar reports, due to Loxton [**28, 29**], Loxton and van der Poorten [**30**], Mahler [**39**], Masser [**47**], Nesterenko [**48**], Ku. Nishioka [**56**], Pellarin

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[**60, 61**], van der Poorten [**66, 67, 68**]. In particular, Ku. Nishioka [**56**] wrote the first and, up to date, the only book entirely devoted to Mahler's method. It provides an invaluable source of information, as well as an exhaustive account up to 1996. The author is indebted to all these mathematicians whose writings helped him a lot to prepare the present survey. He also thanks Michel Waldschmidt for his comments regarding a preliminary version of this text.

2. Introduction

Any algebraic (resp. linear) relation over $\overline{\mathbb{Q}}(z)$ between given analytic functions $f_1(z), \ldots, f_n(z) \in \overline{\mathbb{Q}}\{z\}$[(1)], leads by specialization at a given algebraic point α to an algebraic (resp. linear) relation over $\overline{\mathbb{Q}}$ between the values $f_1(\alpha), \ldots, f_n(\alpha)$, assuming of course that these functions are well-defined at α. The converse is known to be false in general [**39**], but there are few known instances where it holds true. In each case, an additional structure is required: the analytic functions under consideration do satisfy some kind of functional/differential equation. Mahler's method provides an instance of such a phenomenon. In this respect, the proof of Theorem 1.1 is based on the functional equation

$$\mathfrak{f}(z^2) = \mathfrak{f}(z) - z \tag{2.1}$$

which allows one to transfer the presupposed algebraicity of $\mathfrak{f}(\alpha)$ to $\mathfrak{f}(\alpha^{2^n})$, for all integers $n \geq 1$. Furthermore, Theorem 1.1 can be rephrased by saying that the transcendence of the function $\mathfrak{f}(z)$ over $\mathbb{Q}(z)$ is transferred to the transcendence of the values $\mathfrak{f}(\alpha)$ at every non-zero algebraic number in the open unit disc.

As observed by Mahler, an important aspect of his method is that it does not only apply to analytic functions of a single variable and to the operator $z \mapsto z^2$, but also to analytic solutions of different types of functional equations related to more general transformations. Let $\Omega = (t_{i,j})_{1 \leq i,j \leq d}$ be a $d \times d$ matrix with non-negative integer coefficients. We let Ω acts on $\mathbb{C}^d$ by:

$$\Omega \boldsymbol{\alpha} = (\alpha_1^{t_{1,1}} \cdots \alpha_d^{t_{1,d}}, \ldots, \alpha_1^{t_{d,1}} \cdots \alpha_d^{t_{d,d}}),$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d$. We consider as well Ω as acting on monomials $\mathbf{z} = (z_1, \ldots, z_d)$, where $z_1, \ldots, z_d$ are indeterminates. Such an action naturally extends to elements of $\overline{\mathbb{Q}}[[\mathbf{z}]]$ by setting $\Omega f(\mathbf{z}) = f(\Omega \mathbf{z})$. Today Mahler's method encompasses a quite large number of results which makes it not that easy to define. But to sum up, one could reasonably say that:

Mahler's method aims at transferring results about the absence of algebraic or linear relations over $\overline{\mathbb{Q}}(\mathbf{z})$ between analytic solutions of some functional equations related to operators Ω, to their values at suitable algebraic points.

There are three main parameters one has to specify in order to apply Mahler's method:

- a type of functional equation,
- a type of matrix transformation Ω,
- a set of suitable algebraic points $\boldsymbol{\alpha}$.

[(1)] If $\mathbb{K}$ is a subfield of $\mathbb{C}$, $\mathbb{K}\{z\}$ denotes the set of power series with coefficients in $\mathbb{K}$ and which converge in some neighborhood of the origin.

Independently of the choice of the functional equation, there are some unavoidable restrictions that one has to impose on the matrix transformation Ω and on the algebraic point $\boldsymbol{\alpha}$[2]. When these conditions are fulfilled, the pair $(\Omega, \boldsymbol{\alpha})$ is said to be *admissible*. Then there are further restrictions that one has to impose on the point $\boldsymbol{\alpha}$ depending on the functional equation and the matrix Ω. When the latter are fulfilled, the point $\boldsymbol{\alpha}$ is said to be *regular*. With this formalism, all results discussed in the sequel will essentially have the same taste, though difficulties for proving them may be very different. In order to emphasize some unity, we choose to state most results in the sequel as equalities between transcendence degrees. We recall that given a field $\mathbb{K}$ and elements $a_1, \ldots, a_n$ in a field extension of $\mathbb{K}$, the transcendence degree over $\mathbb{K}$ of the field extension $\mathbb{K}(a_1, \ldots, a_n)$, denoted by $\mathrm{tr.deg}_{\mathbb{K}}(a_1, \ldots, a_n)$, is the largest cardinality of an algebraically independent subset of $\mathbb{K}(a_1, \ldots, a_n)$ over $\mathbb{K}$. In particular, saying that $\mathrm{tr.deg}_{\mathbb{K}}(a) = 1$ (resp. $\mathrm{tr.deg}_{\mathbb{K}}(a) = 0$) is equivalent to say that a is transcendental (resp. algebraic) over $\mathbb{K}$.

2.1. Different types of Mahler's equations. — The following three families of equations have been mainly considered so far.

The rational Mahler equation. — It is defined by:

$$f(\Omega \mathbf{z}) = R(\mathbf{z}, f(\mathbf{z})) \,, \tag{2.2}$$

where $R(X,Y) = A(X,Y)/B(X,Y) \in \overline{\mathbb{Q}}(X,Y)$ denotes a two-variate rational function with algebraic coefficients.

The algebraic Mahler equation. — It is defined by:

$$P(\mathbf{z}, f(\mathbf{z}), f(\Omega \mathbf{z})) = 0 \,, \tag{2.3}$$

where $P(\mathbf{z}, X, Y) \in \overline{\mathbb{Q}}[\mathbf{z}, X, Y]$. The rational Mahler equation is a special case of the algebraic Mahler equation where the degree in Y of P is equal to 1.

The linear Mahler equation. — It is defined by:

$$\begin{pmatrix} f_1(\Omega \mathbf{z}) \\ \vdots \\ f_n(\Omega \mathbf{z}) \end{pmatrix} = A(\mathbf{z}) \begin{pmatrix} f_1(\mathbf{z}) \\ \vdots \\ f_n(\mathbf{z}) \end{pmatrix} , \tag{2.4}$$

where $A(\mathbf{z})$ belongs to $\mathrm{GL_n}(\overline{\mathbb{Q}}(\mathbf{z}))$.

Remark 2.1. — It is important to keep in mind that only analytic solutions of the corresponding equations are considered. In particular, Mahler's method does not apply to some interesting functions such as $\log z$ or $\log\log z/\log 2$, though both are solution to simple linear Mahler equations.

2.2. Condition of admissibility. — The existence of a real number $\rho > 1$ such that the following conditions hold seem to be inherent to Mahler's method.

(i) Every entry of the matrix Ω^k belongs to $O(\rho^k)$ as k tends to infinity.

[2] At least when applying Mahler's method in its original form.

(ii) Set $\Omega^k\boldsymbol{\alpha} = (\alpha_1^{(k)}, \ldots, \alpha_d^{(k)})$. Then there exists a positive number c such that for all k large enough:

$$\log|\alpha_i^{(k)}| \leq -c\rho^k\,, \quad 1 \leq i \leq d.$$

(iii) If $f(\mathbf{z})$ is any non-zero element of $\mathbb{C}\{\mathbf{z}\}$, then there are infinitely many integers k such that $f(\Omega^k(\boldsymbol{\alpha})) \neq 0$.

Definition 2.2. — The pair $(\Omega, \boldsymbol{\alpha})$ is said to be *admissible* if conditions (i)–(iii) are fulfilled.

Remark 2.3. — In the case where $d = 1$, the operator Ω takes the simple form $\sigma_q : z \mapsto z^q$, where $q \geq 2$ is an integer, and conditions (i)-(iii) are automatically satisfied with $\rho = q$. The non-vanishing Condition (iii) being just a direct consequence of the identity theorem[(3)].

Our ability to use Mahler's method will thus depend on our ability to provide simple and natural conditions ensuring that $(\Omega, \boldsymbol{\alpha})$ is admissible. Several contributions to this problem are due to Mahler [**36, 37**], Kubota [**27**], Loxton and van der Poorten [**31**], and most notably Masser [**45**]. These works leads finally to the following definitive answer, as described in [**6**].

Definition 2.4. — Let Ω be a $d \times d$ matrix with non-negative integer coefficients and with spectral radius ρ. We say that Ω belongs to the class $\mathcal{M}$ if it satisfies the following three conditions.

(a) It is non-singular.
(b) None of its eigenvalues is a root of unity.
(c) There exists an eigenvector with positive coordinates associated with the eigenvalue ρ.

Definition 2.5. — An algebraic point $\boldsymbol{\alpha} \in (\mathbb{C}^*)^d$ is said to be Ω-*independent* if there is no non-zero d-tuple of integers $\boldsymbol{\mu}$ for which $(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 1$ for all k in an arithmetic progression.

Let us also denote by $\mathcal{U}(\Omega)$ the set of points $\boldsymbol{\alpha}$ of $(\mathbb{C}^*)^d$ such that condition (ii) holds. As observed by Loxton and van der Poorten [**31, 32**], and by Faverjon and the author [**6**], when the matrix Ω belongs to the class $\mathcal{M}$, the set $\mathcal{U}(\Omega)$ is a punctured neighborhood of the origin, and $\mathcal{U}(\Omega) = \left\{\boldsymbol{\alpha} \in (\mathbb{C}^*)^d : \lim_{k\to\infty} \Omega^k\boldsymbol{\alpha} = 0\right\}$. We stress that $\mathcal{U}(\Omega)$ actually contains the punctured open unit disk of $\mathbb{C}^n$. With these definitions, the notion of admissibility can be characterized as follows [**6**].

Theorem 2.6. — *Let Ω be a $d \times d$ matrix with non-negative integer coefficients and $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^*)^d$. Then the pair $(\Omega, \boldsymbol{\alpha})$ is admissible if and only if Ω belongs to the class $\mathcal{M}$, $\lim_{k\to\infty} \Omega^k\boldsymbol{\alpha} = 0$, and $\boldsymbol{\alpha}$ is Ω-independent.*

Theorem 2.6 is essentially a rephrasing of the important Masser vanishing theorem [**45**].

[(3)] That is, the fact that the zeros of a non-zero holomorphic function cannot accumulate inside a connected open set.

2.3. Conditions of regularity. — In addition to the admissibility of the pair $(\Omega, \boldsymbol{\alpha})$, there are still further natural restrictions one has to impose on the point $\boldsymbol{\alpha}$ in order to apply Mahler's method. These restrictions depend both on the functional equation and the matrix transformation Ω.

Before giving them, we first note that the algebraic Mahler Equation (2.3) can be rewrite as

$$A_0(\mathbf{z}, f(\mathbf{z}))f(\Omega\mathbf{z})^r + A_1(\mathbf{z}, f(\mathbf{z}))f(\Omega\mathbf{z})^{r-1} + \cdots + A_r(\mathbf{z}, f(\mathbf{z})) = 0\,,$$

where $A_0 \not\equiv 0$, $A_i(\mathbf{z}, Y) \in \overline{\mathbb{Q}}[\mathbf{z}, Y]$ for $0 \leq i \leq r$, and where the A_i are relatively prime viewed as polynomials in Y. Then there are polynomials $g_i(\mathbf{z}, Y) \in \overline{\mathbb{Q}}[\mathbf{z}, Y]$ such that

$$g(\mathbf{z}) = \sum_{i=0}^{r} g_i(\mathbf{z}, Y)A_i(\mathbf{z}, Y) \tag{2.5}$$

does not depend on $Y$and is not zero.

Definition 2.7. — A point $\boldsymbol{\alpha} \in \mathbb{C}^d$ with non-zero coordinates is said to be *regular*:

- with respect to the rational Mahler Equation (2.2) if $\Delta(\Omega^k\boldsymbol{\alpha}) \neq 0$ for all $k \geq 0$, where Δ denote the resultant of the polynomials A and B viewed as polynomials in Y;
- with respect to the algebraic Mahler Equation (2.3) if $g(\Omega^k\boldsymbol{\alpha}) \neq 0$ for all $k \geq 0$ (where g is defined as in (2.5));
- with respect to linear Mahler Equation (2.4) if the matrix $A(\Omega^k\boldsymbol{\alpha})$ is well-defined and non-singular for all $k \geq 0$.

3. Basic principles of Mahler's method

For the reader who is not familiar with Mahler's method or even with transcendence theory, we first give a proof of Theorem 1.1. The proof is by contradiction, assuming the algebraicity of $\mathfrak{f}(\alpha)$, and follows a scheme of demonstration which is now classical in transcendence theory. The latter can be divided in the following four steps.

(AF)– Constructing an *auxiliary function* together with a corresponding evaluation.

(UB)– Proving an upper bound for this evaluation by means of *analytic estimates.*

(NV)– Proving the non-vanishing of this evaluation by means of a *zero estimates.*

(LB)– Proving a lower bound for this evaluation by means of *arithmetic estimates.*

A contradiction is then derived by comparison between (UB) and (LB) for a suitable choice of underlying parameters. The reader familiar with transcendence theory will notice that each four steps takes here a very primitive form, without requiring any use of Siegel's lemma or of any difficult vanishing theorem. Only simple consideration with heights and the Liouville inequality are needed to achieve this proof. This is certainly an appreciable feature that Mahler's method has.

Remark 3.1. — Note that for the equation $\mathfrak{f}(z^2) = \mathfrak{f}(z) - z$, the transformation matrix $\Omega = (2)$ is just a 1×1 matrix and thus the pair (Ω, α) is admissible for all non-zero algebraic numbers α in the open unit disc. Furthermore, such α are all regular. This explains why Theorem 1.1 applies to *all non-zero algebraic numbers* α with $|\alpha| < 1$.

Height and Liouville's inequaliy. — To prove Mahler's theorem, the only arithmetic tool one needs is a suitable notion of size or complexity that allows one to generalize the fundamental inequality:

$$|p/q| \geq 1/q, \ \forall p/q \in \mathbb{Q}, \ p/q \neq 0$$

to elements of a number field. In order to simplify some computations, it seems more appropriate to use the absolute logarithmic Weil height (or Weil height for short) but any other reasonable notion of height would also do the job. Let $\mathbf{k}$ be a number field. The absolute logarithmic height of a projective point $(\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_n(\mathbf{k})$ is defined by

$$h(\alpha_0 : \cdots : \alpha_n) = \frac{1}{[\mathbf{k} : \mathbb{Q}]} \sum_{\nu \in M_{\mathbf{k}}} d_\nu \log \max\{|\alpha_0|_\nu, \ldots, |\alpha_n|_\nu\} ,$$

where ν runs over a complete set $M_{\mathbf{k}}$ of non-equivalent places of $\mathbf{k}$, $d_\nu = [\mathbf{k}_\nu : \mathbb{Q}_p]$(4), and where the absolute values $|\cdot|_\nu$ are normalized so that the product formula holds:

$$\prod_{\nu \in M_{\mathbf{k}}} |x|_\nu^{d_\nu} = 1 , \ \forall x \neq 0 \in \mathbf{k} ,$$

We also set $h(\alpha) = h(1 : \alpha)$ for all $\alpha \in \mathbf{k}$, so that $h(0) = 0$. We stress that the height of an algebraic number α does not depend on the choice of the number field $\mathbf{k}$ containing it. The following useful properties can be easily derived from the definition. Given two algebraic numbers α and β, one has:

$$\begin{array}{rcl} h(\alpha+\beta) & \leq & h(\alpha) + h(\beta) + \log 2 \\ h(\alpha\beta) & \leq & h(\alpha) + h(\beta) \\ h(\alpha^n) & = & nh(\alpha), \quad n \in \mathbb{N} \\ h(1/\alpha) & = & h(\alpha), \quad \alpha \neq 0. \end{array} \tag{3.1}$$

More generally, if $P \in \mathbb{Z}[X_1, \ldots, X_n]$ and $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$, one has

$$h(P(\alpha_1, \ldots, \alpha_n)) \leq \log L(P) + \sum_{i=1}^{n} (\deg_{X_i} P) h(\alpha_i) , \tag{3.2}$$

where $L(P)$ denotes the length of P, that is the sum of the absolute values of its coefficients. As a direct consequence of the product formula, one also obtains the fundamental *Liouville inequality*:

$$\log|\alpha| \geq -[\mathbf{k} : \mathbb{Q}] h(\alpha), \ \forall \alpha \neq 0 \in \mathbf{k} . \tag{3.3}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. — Let us assume by contradiction that α and $\mathfrak{f}(\alpha)$ are both algebraic. Let $\mathbf{k}$ be a number field containing these numbers and let N be a positive integer.

(AF) The first observation is that there exist $N+1$ polynomials $P_0(z), \ldots, P_N(z) \in \mathbb{Z}[z]$ of degree at most N, not all zero, such that the power series

$$E_N(z) = \sum_{j=0}^{N} P_j(z) \mathfrak{f}^j(z) = \sum_{h=0}^{\infty} e_h z^h$$

(4) Here $\mathbf{k}_\nu$ denote the completion of $\mathbf{k}$ with respect to ν and $\nu|_{\mathbb{Q}} = p$, with the convention that if $\nu|_{\mathbb{Q}} = \infty$ then $\mathbb{Q}_p = \mathbb{R}$ and $\mathbf{k}_\nu$ is either $\mathbb{R}$ if the place is real or $\mathbb{C}$ if the place is complex.

has valuation at least N^2+1 at zero, that is $e_h = 0$ for all $h \leq N^2$. Indeed finding such polynomials is equivalent to solve a linear system with $(N+1)^2$ unknowns (the coefficients of the polynomials P_j) and only N^2+1 equations.

(UP) The previous construction ensures that $E_N(z)$ takes small values around zero. More precisely, one easily gets that

$$|E_N(\alpha^{2^n})| \leq c_1(N)|\alpha|^{2^n N^2}, \tag{3.4}$$

where $c_1(N)$ depends on N but not on n.

(NV) We now observe that $f(z)$ is a transcendental function over $\mathbb{Q}(z)$. This follows for instance from the fact that its Taylor series expansion is too sparse. In consequence, our auxiliary function $E_N(z)$ is a non-zero analytic function, which implies by the identity theorem that $E_N(\alpha^{2^n})$ is non-zero for n large enough.

(LB) Equation (2.1) ensures that all numbers $E_N(\alpha^{2^n})$ belong to $\mathbf{k}$. Let us denote by $A_N(X,Y) \in \mathbf{k}[X,Y]$ the polynomial, of degree at most N in X and Y, such that

$$A_N(z,\mathfrak{f}(z)) = \sum_{j=0}^{N} P_j(z)\mathfrak{f}^j(z).$$

Thus $E_N(\alpha^{2^n}) = A_N(\alpha^{2^n},\mathfrak{f}(\alpha^{2^n}))$ and a routine calculation using (3.1) and (3.2) gives that

$$h(E_N(\alpha^{2^n})) \leq c_2(N) + 2^{n+1}Nh(\alpha) + Nn,$$

where $c_2(N)$ depends on N but not on n. We then infer from Liouville's Inequality (3.3) that

$$\log|E_N(\alpha^{2^n})| \geq -[\mathbf{k}:\mathbb{Q}]\left(c_2(N) + 2^{n+1}Nh(\alpha) + Nn\right). \tag{3.5}$$

We are now ready to conclude the proof. By (3.4) and (3.5), we get that

$$c_1(N) + 2^n N^2 \log|\alpha| \geq -[\mathbf{k}:\mathbb{Q}](c_2(N) + 2^{n+1}Nh(\alpha) + Nn).$$

Dividing by 2^n and letting n tend to infinity, it follows that

$$N \leq \frac{2[\mathbf{k}:\mathbb{Q}]h(\alpha)}{|\log|\alpha||}.$$

The latter inequality provides a contradiction for N can be chosen arbitrarily large. $\square$

4. The rational Mahler equation

Using essentially the same arguments as in the proof of Theorem 1.1, Mahler [**36, 37**] was already able to prove the following general result.

Theorem 4.1. — *Let $\mathbf{k}$ be a number field and let us assume that $f(\mathbf{z}) \in \mathbf{k}\{\mathbf{z}\}$ is solution to Equation* (2.2). *Let m denote the maximal degree in Y of the denominator and the numerator of $R(X,Y)$ and let us assume that $m < \rho$, the spectral radius of Ω. If the pair $(\Omega,\boldsymbol{\alpha})$ is admissible and $\boldsymbol{\alpha}$ is regular, then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}} f(\boldsymbol{\alpha}) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\mathbf{z})} f(\mathbf{z}).$$

Let us make few comments.

• The assumption $m < \rho$ is used in the proof to get suitable estimates for heights in step (LB). However, Ku. Nishioka [**51**] has shown how to relax this assumption to $m < q^2$ in the case where $d = 1$ and $\Omega = (q)$. The new argument provided by Nishioka is the use of Siegel's lemma in order to take care about the size of the coefficients of the polynomials P_j in step (AF), instead of just solving a linear system.

• The assumption about regularity cannot be removed as shown by the following simple example. Let

$$\mathfrak{g}_1(z) = \prod_{n=0}^{\infty}(1 - 2z^{2^n})$$

that satisfies the equation $\mathfrak{g}_1(z^2) = \mathfrak{g}_1(z)/(1-2z)$. One can check that $\mathfrak{g}_1(z)$ is transcendental but $\mathfrak{g}_1(\alpha) = 0$ for all α such that $\alpha^{2^n} = 1/2$ for some n. Here, $\Delta(z) = (1-2z)$, and $\Delta(\alpha^{2^n}) = 0$ for such α. If one slightly modified the definition of $\mathfrak{g}_1(z)$ to

$$\mathfrak{g}_2(z) = \prod_{n=0}^{\infty}(1 - z^{2^n})$$

then $\Delta(z) = (1 - z)$ has no root inside the open unit circle, and it follows that $\mathfrak{g}_2(\alpha)$ is transcendental for all non-zero algebraic numbers α with $|\alpha| < 1$.

• The transcendence of the function $f(\mathbf{z})$ in Theorem 4.1 is of course necessary to obtain the transcendence of the value $f(\boldsymbol{\alpha})$. Let us recall the following informative anecdote. As an application of his theorem, Mahler considered in [**41**] the function

$$\mathfrak{h}(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^{n+1}}} \cdot$$

It satisfies the simple equation $\mathfrak{h}(z^2) = \mathfrak{h}(z) - z/(1 - z^2)$, from which he deduced the transcendence of the number

$$2 - \mathfrak{h}((1 - \sqrt{5})/2) = \sum_{n=0}^{\infty} \frac{1}{F_{2^n}} ,$$

where F_n denotes the n-th Fibonacci number. But in fact the latter series is equal to $(7 - \sqrt{5})/2$. The reason for this mistake is just that Mahler forgot to check that $\mathfrak{h}(z)$ is transcendental and it turns out that this is not the case for $\mathfrak{h}(z) = z/(1 - z)$. It is worth mentioning that the transcendance of analytic solutions to the rational Mahler equation can generally be deduced from the following useful dichotomy due to Keiji Nishioka [**50**]: a power series $f(\mathbf{z})$ satisfying (2.2) either belongs to $\mathbf{k}(\mathbf{z})$ or is transcendental over $\mathbf{k}(\mathbf{z})$.

One Mahler favorite consequence of Theorem 4.1 was the following.

Corollary 4.2. — *Let ω be a quadratic irrational real number and set*

$$\mathfrak{f}_\omega(z) = \sum_{n=0}^{\infty} \lfloor n\omega \rfloor z^n .$$

Then $\mathfrak{f}_\omega(\alpha)$ is transcendental for all algebraic numbers α with $0 < |\alpha| < 1$.

In order to prove Corollary 4.2, Mahler first considered the bivariate function

$$F_\omega(z_1, z_2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\lfloor n\omega \rfloor} z_1^n z_2^m$$

and used the theory of continued fractions to prove that it satisfies the following equation:

$$F_\omega(z_1,z_2)=\sum_{k=0}^{t-1}(-1)^k\frac{z_1^{p_{k+1}+p_k}z_2^{q_{k+1}+q_k}}{\left(1-z_1^{p_{k+1}}z_2^{q_{k+1}}\right)\left(1-z_1^{p_k}z_2^{q_k}\right)}+F_\omega(\Omega(z_1,z_2))$$

where

$$\Omega=\begin{pmatrix} p_t & q_t \\ p_{t-1} & q_{t-1}\end{pmatrix}.$$

Here, p_n/q_n denotes the n-th convergent to ω, and ω is assumed to have a purely periodic continued fraction expansion of even period t[5]. Then Mahler deduced from Theorem 4.1 that $F_\omega(\alpha,1)=\mathfrak{f}_\omega(\alpha)$ is transcendental for all algebraic number α with $0<|\alpha|<1$.

Remark 4.3. — This example well-illustrated the relevance of working with multivariate functions. Indeed, it is expected that the one-variable function $\mathfrak{f}_\omega(z)$ does not satisfy any one-variable Mahler equation, but $\mathfrak{f}_\omega(z)=F_\omega(z,1)$ can be obtained as the specialization of a two-variate Mahler function.

The interested reader will find stronger results about algebraic independence of values at algebraic points of the so-called Hecke–Mahler series $\mathfrak{f}_\omega(z)$ in [**7, 46, 59**].

5. The algebraic Mahler equation

In his three papers from 1929–30, Mahler did not consider the general algebraic equation (2.3). In fact, he never proved any transcendence result concerning this equation. However, he explicitly mentioned it at several places [**39, 40**] and Problem 3 in [**40**] reads as follows.

Problem 5.1. — *Assume that $f(\mathbf{z})\in\overline{\mathbb{Q}}\{\mathbf{z}\}$ satisfies the equation*

$$P(\mathbf{z},f(\mathbf{z}),f(\Omega\mathbf{z}))=0\,,$$

where $P(\mathbf{z},X,Y)\in\overline{\mathbb{Q}}[\mathbf{z},X,Y]$ is non-zero. To investigate the transcendency of function values $f(\mathbf{z}_0)$ where $\mathbf{z}_0$ is an algebraic point satisfying suitable further restrictions.

When raising Problem 5.1, Mahler was motivated by the study of the values of the elliptic modular invariant $j(\tau)$ at algebraic points τ. A classical result of Schneider from 1937 shows that $j(\tau)$ is transcendental, unless τ is a quadratic number. Setting $j(\tau)=:J(e^{2i\pi\tau})$, the function $J(z)$ is meromorphic in the open unit disc and $J(z)-1/z$ has a power series expansion with non-negative integer coefficients

$$J(z)-\frac{1}{z}=744+196\ 884\ z+21\ 493\ 760\ z^2+\cdots.$$

Mahler conjectured that $J(\alpha)$ is transcendental for all algebraic α with $0<|\alpha|<1$. His hope was to prove this conjecture by developing a suitable extension of his method to the general equation (2.3). Indeed, it is known that for all positive integers q, there exists a polynomial $\Phi_q(X,Y)\in\mathbb{Z}[X,Y]$ such that

$$\Phi_q(J(z),J(z^q))=0\,.$$

The main contribution to Problem 5.1 is due to Ku. Nishioka [**51, 52**]. In order to avoid heavy notation, we only state it in the one-dimensional case, that is when $\Omega=(q)$.

[5] Mahler also explained how to reduce to this particular case.

Theorem 5.2. — *Let* $\mathbf{k}$ *be a number field and let us assume that* $f(z) \in \mathbf{k}\{z\}$ *is solution of Equation* (2.3)*. Let us furthermore assume that* $mn^2 < q^2$ *where* m *(resp.* n*) denotes the degree of* $P(z, X, Y)$ *in the variable* X *(resp.* Y*). If* α *is regular, then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}} f(\alpha) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)} f(z)\,.$$

Unfortunately, the modular invariant J remains beyond the scope of Mahler's method for the condition $mn^2 < q^2$ is not fulfilled. Indeed, the degree n in Y of Φ_q is larger than q. Nevertheless, Mahler's ideas were really influential and in 1996, his conjecture was finally confirmed by the *Stephanese team*: Barré-Sirieix, Diaz, Gramain, Philibert [**11**]. Strictly speaking, the approach used by these authors is not Mahler's method, but it is somewhat reminiscent of it.

6. The linear Mahler equation

Despite its natural analogy with linear differential equations and the theory of Siegel's E-functions, it is rather surprising that Mahler never really considered what is now referred to as the linear Mahler equation. In contrast, the linear Mahler equation has taken on more and more importance over the years and most current researches on Mahler's method focus now on such equations. Indeed, during the late Seventies, Mahler's method really took on a new significance after Mendès France popularized the fact that linear Mahler equations naturally arise in the study of automata theory. Though already explicitly noticed in 1968 by Cobham [**19**], this connection remained relatively unknown at that time, likely because Cobham's works on this topic were not published in academic mathematical journals. In section 7, we will briefly discuss this important application of Mahler's method.

6.1. The one-variable case. — Given an integer $q \geq 2$, a function $f(z) \in \overline{\mathbb{Q}}\{z\}$ is said to be a *q-Mahler function* if there exist polynomials $p_0(z), \ldots, p_n(z) \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_n(z)f(z^{q^n}) \;=\; 0\,. \tag{6.1}$$

A function $f(z) \in \overline{\mathbb{Q}}\{z\}$ is q-Mahler if, and only if, $f(z)$ is a coordinate of a vector solution to a one-variable linear Mahler system:

$$\begin{pmatrix} f_1(z^q) \\ \vdots \\ f_n(z^q) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}, \tag{6.2}$$

where $A(z)$ is a matrix in $\mathrm{GL_n}(\overline{\mathbb{Q}}(\mathrm{z}))$ and the f_i's are analytic in a neighborhood of $z = 0$. We will also simply say that $f(z)$ is a *M-function* if it is q-Mahler for some q. An M-function is always meromorphic on the open unit disc, the unit circle being a natural boundary unless if it is a rational function [**69**]. In particular, as for solutions to the rational Mahler equation, an M-function is either rational or transcendental over $\overline{\mathbb{Q}}(z)$. We also recall that the Taylor coefficients of an M-function are always confined in some number field.

The above definitions highlight a strong analogy with the theory of E- and G-fonctions introduced by Siegel: linear differential equations are replaced by linear difference equations associated with the Mahler operator $\sigma_q : z \mapsto z^q$. The main results concerning M-functions turn out to be in complete correspondence with those obtained for E-fonctions. After several partial results due to Mahler [**38**], Kubota [**27**], Loxton and van der Poorten [**34, 35**], the analog of the Siegel–Shidlovskii theorem was finally obtained by Ku. Nishioka [**53**] in 1990.

Theorem 6.1. — *Let $f_1(z),\dots,f_n(z)\in\overline{\mathbb{Q}}\{z\}$ be solutions to (6.2). Let $\alpha\in\overline{\mathbb{Q}}$, $0<|\alpha|<1$, be a regular point with respect to this system. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha),\dots,f_n(\alpha))=\mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\dots,f_n(z))\,.$$

The proof of Nishioka is based on some technics from commutative algebra introduced in the framework of algebraic independence by Nesterenko in the late Seventies. Recently, Fernandes observed that Theorem 6.1 can also be deduced from a general algebraic independence criterion due to Philippon [**62**](6).

It is quite tempting to believe that M-functions should take transcendental values at non-zero algebraic points in the open unit disc. In some sense, this is the case but there may be some exceptions, as illustrated by the example of the function $\mathfrak{g}_1(z)$ given after Theorem 4.1. Theorem 6.1 implies the following simple dichotomy: a transcendental M-function solution to a linear scalar Mahler equation of order one (possibly inhomogeneous) takes algebraic values(7) at singular points and transcendental values at all other algebraic points. However, as powerful as it is, Nishioka's theorem does not completely solve the question of the algebraicity/transcendence for the values at algebraic points of M-functions satisfying higher order equations. There are two reasons for that. First, in the general case, the transcendence of the function $f_1(z)$ does not ensure that the number $f_1(\alpha)$ is transcendental, but only that one among the numbers $f_1(\alpha),\dots,f_n(\alpha)$ is transcendental, assuming furthermore that α is a regular point. The second difficulty arises precisely from the fact that Nishioka's Theorem does not apply at singular points.

In 2006, Beukers [**14**] obtained a refined version of the Siegel–Shidlovskii theorem as a consequence of the work of André [**9**] on E-operators. Another proof of this beautiful result is given by André in [**10**]. Inspired by these works, and by the work of Nesterenko and Shidlovskii [**49**], a similar refinement for linear Mahler systems has been proved recently by Philippon [**65**]. The following stronger homogeneous version is obtained in [**4**]. It is the exact analog of Beuker's theorem.

Theorem 6.2. — *Let $f_1(z),\dots,f_n(z)\in\overline{\mathbb{Q}}\{z\}$ be solutions to (6.2). Let $\alpha\in\overline{\mathbb{Q}}$, $0<|\alpha|<1$, be a regular point for this system. Then for all homogeneous polynomial $P\in\overline{\mathbb{Q}}[X_1,\dots,X_n]$ such that $P(f_1(\alpha),\dots,f_n(\alpha))=0$, there exists $Q\in\overline{\mathbb{Q}}(z)[X_1,\dots,X_n]$, homogeneous in $X_1,\dots,X_n$, such that $Q(z,f_1(z),\dots,f_n(z))=0$ and $Q(\alpha,X_1,\dots,X_n)=P(X_1,\dots,X_n)$.*

The problem of the transcendence, and more generally of the linear independence over $\overline{\mathbb{Q}}$, of values of M-functions at algebraic points is completely solved in [**4, 5**] by using Theorem 6.2. In this direction, we quote Theorem 6.3 below. The main feature of this result is that it applies to *all non-zero algebraic points* in the open unit disc and not only to regular points. This is a crucial point for applications such as described in Section 7.

(6) Note that the proof of this algebraic independence criterion also relies on the method introduced by Nesterenko.

(7) In fact, such an M-function necessarily vanishes at all non-zero singular points in the homogeneous case.

Theorem 6.3. — *Let* $\mathbf{k}$ *be a number field,* $f(z) \in \mathbf{k}\{z\}$ *be an* M*-function, and* α *be an algebraic number,* $0 < |\alpha| < 1$*, that is not a pole of* $f(z)$*. Then either* $f(\alpha)$ *is transcendental or* $f(\alpha) \in \mathbf{k}(\alpha)$*. Furthermore there exists an algorithm to decide this alternative.*

One can reasonably argue that Theorem 6.2 provides essentially all what transcendence theory has to say about the algebraic relations between the values of several q-Mahler functions at a given algebraic point. In contrast, the previous results do not say that much about the algebraic relations between the values of several M-functions (possibly associated with different Mahler operators) at *distinct* algebraic points. In this direction, we mention the following general conjecture [**7**]. We recall that given complex numbers $\alpha_1, \ldots, \alpha_r$ are said to be *multiplicatively independent* if there is no non-zero tuple of integers $n_1, \ldots, n_r$ such that $\alpha_1^{n_1} \cdots \alpha_r^{n_r} = 1$.

Conjecture 6.4. — *Let* $r \geq 2$ *be an integer. For every integer* i*,* $1 \leq i \leq r$*, we let* $q_i \geq 2$ *be an integer,* $f_i(z) \in \overline{\mathbb{Q}}\{z\}$ *be a* q_i*-Mahler function, and* α_i *be an algebraic number,* $0 < |\alpha_i| < 1$*, such that* $f_i(z)$ *is well-defined at* α_i*. Then the following properties hold.*

(i) *Let us assume that* $\alpha_1, \ldots, \alpha_r$ *are multiplicatively independent. Then the numbers* $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ *are algebraically independent over* $\overline{\mathbb{Q}}$ *if and only if they are all transcendental.*

(ii) *Let us assume that* $q_1, \ldots, q_r$ *are pairwise multiplicatively independent. Then the numbers* $f_1(\alpha_1), f_2(\alpha_2), \ldots, f_r(\alpha_r)$ *are algebraically independent over* $\overline{\mathbb{Q}}$ *if and only if they are all transcendental.*

The main result of [**7**] provides a significant progress toward this conjecture (see Section 6.2).

6.2. The multivariate case. — All results that have been obtained so far concerning multivariate linear Mahler systems (2.4) are restricted to the so-called *regular singular* case.

Definition 6.5. — For every integer $\ell \geq 1$, we let $\widehat{\mathbf{K}}_\ell$ denote the field of fractions of $\overline{\mathbb{Q}}\{\mathbf{z}^{1/\ell}\}$, where $\mathbf{z}^{1/\ell} = (z_1^{1/\ell}, \ldots, z_d^{1/\ell})$. We also set

$$\widehat{\mathbf{K}} := \cup_{\ell \geq 1} \widehat{\mathbf{K}}_\ell\,.$$

A linear Mahler system of type (2.4) is said to be regular singular if there exists a matrix $\Phi(\mathbf{z}) \in \mathrm{GL}_n(\widehat{\mathbf{K}})$ such that $\Phi(T\mathbf{z})A(\mathbf{z})\Phi^{-1}(\mathbf{z}) \in \mathrm{GL}_n(\overline{\mathbb{Q}})$.

This means that there exists a meromorphic (possibly ramified) gauge transform that changes the initial system into a system associated with a matrix with constant coefficients. As shown in [**34**], this is in particular the case when $A(\mathbf{0})$ is well-defined and non-singular. The following multivariate analog of Theorem 6.2 was recently announced in [**6**].

Theorem 6.6. — *Let us consider a regular singular system of type* (2.4) *for which the pair* $(\Omega, \boldsymbol{\alpha})$ *is admissible and such that* $\boldsymbol{\alpha}$ *is regular. Then for all homogeneous* $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ *such that* $P(f_1(\boldsymbol{\alpha}), \ldots, f_n(\boldsymbol{\alpha})) = 0$*, there exists* $Q \in \overline{\mathbb{Q}}(\mathbf{z})[X_1, \ldots, X_n]$*, homogeneous in* $X_1, \ldots, X_n$*, such that* $Q(\mathbf{z}, f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})) = 0$ *and* $Q(\boldsymbol{\alpha}, X_1, \ldots, X_n) = P(X_1, \ldots, X_n)$*.*

In 1982, Loxton and van der Poorten [**34**] published a paper claiming the weaker result

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\boldsymbol{\alpha}), \ldots, f_n(\boldsymbol{\alpha})) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(\mathbf{z}), \ldots, f_n(\mathbf{z})) \tag{6.3}$$

under the assumptions of Theorem 6.6. Unfortunately, some steps of their proof was flawed as was noticed for instance by Ku. Nishioka in [**53**]. We stress that before [**6**], Equality (6.3) was only proved in two very restrictive cases. First, in the almost diagonal case, that is when each function $f_i(\mathbf{z})$ satisfies an equation of the form:

$$f_i(\mathbf{z}) = a_i(\mathbf{z}) f_i(T.\mathbf{z}) + b_i(\mathbf{z}), \qquad a_i(\mathbf{z}), b_i(\mathbf{z}) \in \overline{\mathbb{Q}}(z),\ a_i(\mathbf{0}) \neq 0\,, \tag{6.4}$$

by Kubota [**27**] in 1977, as a generalization of a theorem of Mahler [**38**]. And then in the almost constant case by Ku. Nishioka [**55**] in 1996, that is for systems of the form:

$$\begin{pmatrix} f_1(\Omega\mathbf{z}) \\ \vdots \\ f_m(\Omega\mathbf{z}) \\ 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & & & b_1(\mathbf{z}) \\ & A & & \vdots \\ & & & b_m(\mathbf{z}) \\ \hline 0 & \dots & 0 & 1 \end{array}\right) \begin{pmatrix} f_1(\mathbf{z}) \\ \vdots \\ f_m(\mathbf{z}) \\ 1 \end{pmatrix},$$

where $A \in \mathrm{GL_n}(\overline{\mathbb{Q}})$ and $b_1(\mathbf{z}), \dots, b_m(\mathbf{z})$ are rational functions.

Remark 6.7. — One of the main interest of a result like Theorem 6.6 is that it can be used to prove the algebraic independence for values of a single univariate function at distinct algebraic points. For instance, with the function $\mathfrak{f}(z) = \sum_{n=0}^{\infty} z^{2^n}$ of the introduction, one can associate the linear Mahler system

$$\begin{pmatrix} 1 \\ \mathfrak{f}(z_1^2) \\ \mathfrak{f}(z_2^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -z_1 & 1 & 0 \\ -z_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathfrak{f}(z_1) \\ \mathfrak{f}(z_2) \end{pmatrix}.$$

The underlying transformation matrix

$$\Omega = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

belongs to the class $\mathcal{M}$. Furthermore, the point $\boldsymbol{\alpha} = (1/2, 1/3)$ is regular and the pair $(\Omega, \boldsymbol{\alpha})$ is admissible. From the transcendence of $\mathfrak{f}(z)$, one then gets for free the algebraic independence over $\overline{\mathbb{Q}}(z_1, z_2)$ of $\mathfrak{f}(z_1)$ and $\mathfrak{f}(z_2)$. Theorem 6.6 thus ensures that the real numbers $\mathfrak{f}(1/2)$ and $\mathfrak{f}(1/3)$ are algebraically independent over $\overline{\mathbb{Q}}$. Such applications make Mahler's method in several variables a powerful tool for the study of algebraic independence of automatic numbers which are discussed in Section 7.

To conclude this section, let us mention one other important new feature in [**6**]. Two linear Mahler systems $\mathbf{f}_1(\Omega_1\mathbf{z}) = A_1(\mathbf{z})\mathbf{f}_1(\mathbf{z})$ and $\mathbf{f}_2(\Omega_2\mathbf{z}) = A_2(\mathbf{z})\mathbf{f}_1(\mathbf{z})$ can always be artificially glued together as

$$\begin{pmatrix} \mathbf{f}_1(\Omega\mathbf{z}_1) \\ \mathbf{f}_2(\Omega\mathbf{z}_2) \end{pmatrix} = \begin{pmatrix} A_1(\mathbf{z}_1) & 0 \\ 0 & A_2(\mathbf{z}_2) \end{pmatrix} \begin{pmatrix} \mathbf{f}_1(\mathbf{z}_1) \\ \mathbf{f}_2(\mathbf{z}_2) \end{pmatrix},$$

where

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}.$$

When the transformation matrices Ω_1 and Ω_2 belong to the class $\mathcal{M}$ and have the same spectral radius, then Ω again belongs the class $\mathcal{M}$, so that Theorem 6.6 applies. But this is no longer true when Ω_1 and Ω_2 have multiplicatively independent spectral radii. In [**6**], Mahler's method is generalized in such a way that it also covers this case which is of particular importance for applications to automata theory. Such a generalization was first envisaged by Loxton and van der Poorten [**68**], but only very partial results had been proved so far [**27, 54**].

A q-Mahler function $f(z)$ is said to be regular singular if it is the coordinate of a vector representing a solution to a regular singular Mahler system of the form (6.2). As a consequence of the general results obtained in [**6**] concerning regular singular Mahler systems in several variables, Faverjon and the author [**7**] prove that Conjecture 6.4 is true if each function $f_i(z)$ is assumed to be regular singular.

7. Application to the study of integer base expansions of real numbers

An old source of frustration for mathematicians arises from the study of integer base expansions of classical constants like

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\cdots$$

or

$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\,502\,884\,197\cdots$$

While these numbers admit very simple geometric descriptions, a close look at their digital expansions suggest highly complex phenomena. Over the years, different ways have been envisaged to formalize this old problem. Each of these points of view leads to a different assortment of challenging conjectures. In 1965, Hartmanis and Stearns [**26**] investigated the fundamental question of how hard a real number may be to compute, introducing the now classical time complexity classes. The notion of time complexity takes into account the number of elementary operations needed by a multitape deterministic Turing machine to produce the first n digits of the expansion. In this regard, a real number is considered all the more simple as its base-b expansion can be produced very fast by a Turing machine. At the end of their paper, Hartmanis and Stearns suggested the following problem which is still widely open. We refer the reader to [**3**] for a detailed discussion on Problem 7.1.

Problem 7.1. — *Do there exist irrational algebraic numbers for which the first n binary digits can be computed in $O(n)$ operations by a multitape deterministic Turing machine?*

In 1968, Cobham [**19**] suggested to restrict the Hartmanis-Stearns problem to some simple classes of Turing machines. The main model of computation he investigated is the so-called *Finite automaton*. Such devices can be used to output the well-known *automatic sequences*. We refer the reader to the monograph [**8**] for formal definitions about these notions.

7.1. Cobham's "Theorem" + Corollary. — At this point, it is worth mentioning that Cobham was likely unaware of the existence of Mahler's works. Even so, he was the first to understand that transcendence results for the values of M-functions at algebraic points would have nice consequences to the Hartmanis–Stearns problem. The analogy between M-functions and E-functions pushed him to claim in [**19**] that the following statement should be true. But he never provided a proof, even a sketch of it.

"Theorem". — *Let $f_1(z),\ldots,f_n(z) \in \mathbb{Q}\{z\}$ be solutions to the linear Mahler system (6.2) which are analytic in the open unit disc. Let α be a rational number with $0 < |\alpha| < 1$. Then for all $\lambda_1,\ldots,\lambda_n \in \mathbb{Q}$, the number*

$$\lambda_1 f_1(\alpha) + \cdots + \lambda_n f_n(\alpha)$$

is either rational or transcendental.

This result is a direct consequence of Theorem 6.3 for q-Mahler functions with rational coefficients form a ring. In addition, Cobham [**19**] proved that generating functions associated with automatic sequences do satisfy linear Mahler equations; thus providing a bridge between Mahler's method and the study of the complexity of sequences of digits of real numbers. In particular, he proved that his "theorem" implies the following corollary.

Corollary 7.2 **(Cobham's and/or Loxton–van der Poorten's conjecture.)**
The base-b expansion of an algebraic irrational real number cannot be generated by a finite automaton.

This nice application of transcendence theory to the study of the computational complexity of algebraic numbers had served as an important source of motivation for developing Mahler's method from the late Seventies. Several authors, including especially Loxton and van der Poorten [**29, 34, 35**], have then tried to prove Theorem 6.1, until Nishioka finally proved it. Loxton and van der Poorten also claimed that Corollary 7.2 would follow from this result. This is the reason why Corollary 7.2 was sometimes referred to as the Loxton–van der Poorten conjecture despite the fact it was first suggested by Cobham in 1968. Unfortunately, there are two major obstructions that prevent Theorem 6.1 to imply Corollary 7.2 as explained in Section 6.1. The Loxton–van der Poorten conjecture was finally proved by Bugeaud and the author in [**2**] by a totally different approach based on the p-adic subspace theorem. But this is only after the recent work of Philippon [**65**], that Cobham's "Theorem" was fully proved in [**4**], solving thus a long-standing problem in Mahler's method.

As discussed in [**7**], Conjecture 6.4 would have important consequences concerning *automatic real numbers*, that is real numbers whose base-b expansion can be generated by a finite automaton for some integer base b.

8. Application to transcendence of periods in characteristic p

In this section we briefly discuss an aspect of Mahler's method that would likely deserve to be better known. It is concerned with transcendence and algebraic independence over function fields of positive characteristic. To sum-up, one could roughly say that Mahler's method is *free of characteristic*. Of course, we will not try to justify this informal claim but instead we will content ourself with a brief exposition of a typical situation, essentially reproduced from [**24**]. We refer the interested reader to the papers of Pellarin [**60, 61**] for nice surveys about Mahler's method in characteristic p. We first recall in Figure 1 some fundamental analogies between number fields and function fields of positive characteristic.

The field L either denotes a number field or a function field of characteristic p, while, in each framework, the field C is both complete and algebraically closed. These fundamental analogies allow one to translate many classical number theoretical problems from number fields to function fields in characteristic p, and in particular some which are related to transcendence and algebraic independence. As mentioned in Section 6.1, Fernandes [**24**] recently shows that Nishioka's theorem (Theorem 6.1) can in fact be deduced from a general algebraic independence criterion due to Philippon. This criterion applies in a great generality [**63**], which covers in particular both frameworks mentioned above. As a consequence, she obtains the following statement that is equally valid in one or the other context.

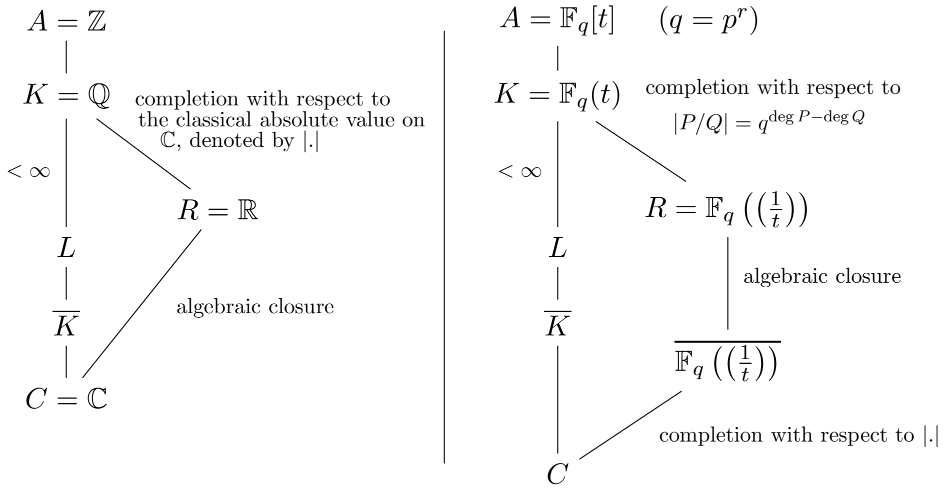

FIGURE 1. Analogies between number fields and function fields in characteristic p.

Theorem 8.1. — *Let $f_1(z), \ldots, f_n(z) \in \overline{K}\{z\}$ be solutions to the linear Mahler system (6.2). Let $\alpha \in \overline{K}$, $0 < |\alpha| < 1$, be a regular point with respect to this system. Then*

$$\mathrm{tr.deg}_{\overline{K}}(f_1(\alpha), \ldots, f_n(\alpha)) = \mathrm{tr.deg}_{\overline{K}(z)}(f_1(z), \ldots, f_n(z)) .$$

When $K = \mathbb{Q}$, this is precisely Nishioka's theorem. It is also worth mentioning that Denis [**20**] first proved Theorem 8.1 for a special class of Mahler's systems when $K = \mathbb{F}_q(t)$. He was motivated by the following remarkable discovery on his own: some analogs of periods in this framework (such as analogs of the number π or of integer values of the Riemann ζ function) can be obtained as values at algebraic points of p-Mahler functions, where p denotes the characteristic of the ground field under consideration. This makes Mahler's method a powerful tool for the study of transcendence and algebraic independence of periods of Drinfeld modules (or more generally of t-modules). The following example well illustrates this phenomenon. In analogy with the Riemann zeta function, we define the Carlitz zeta function by

$$\zeta_C(s) = \sum_{a \in \mathbb{F}_q[t],\ a \text{ unitary}} \frac{1}{a^s} \, .$$

The following formula was proved by Carlitz in [**17**]:

$$\zeta_C(s) = \sum_{h=0}^{\infty} \frac{(-1)^{hs}}{(L_h)^s} \, ,$$

for every integer s, $1 \leq s \leq p-1$, where

$$L_h = \left(t^{q^h} - t\right)\left(t^{q^{h-1}} - t\right) \cdots \left(t^q - t\right) \, ,$$

for all $h \geq 1$, and $L_0 = 1$. The key point is that the number $\zeta_C(s)$ can be deformed to construct a p-Mahler function

$$f_s(z) = \sum_{h=0}^{\infty} \frac{(-1)^{hs}}{\left(\left(z^{q^h} - t\right)\left(z^{q^{h-1}} - t\right) \cdots \left(z^q - t\right)\right)^s} \, .$$

Furthermore, by construction one has:

$$f_s(t) = \zeta_C(s)\,.$$

It is easy to see that the Mahler equation satisfied by $f_s(z)$ is just

$$f_s(z^q) = (-1)^s\,(z^q - t)^s\, f_s(z) + (-1)^{s+1}\,(z^q - t)^s\ .$$

Now, we can put these individual equations into the following linear Mahler system:

$$\begin{pmatrix} 1 \\ f_1(z^q) \\ \vdots \\ f_{p-1}(z^q) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ z^q - t & -(z^q-t) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ (-1)^p\,(z^q-t)^{p-1} & & & & (-1)^{p-1}\,(z^q-t)^{p-1} \end{pmatrix} \begin{pmatrix} 1 \\ f_1(z) \\ \vdots \\ f_{p-1}(z) \end{pmatrix}.$$

By Theorem 8.1, the algebraic independence over $\mathbb{F}_q(t)$ of the periods

$$\zeta_C(1), \zeta_C(2), \ldots, \zeta_C(p-1)\,,$$

will then follow from the algebraic independence of the functions $f_1(z), f_2(z), \ldots, f_{p-1}(z)$ over $\mathbb{F}_q(t)(z)$. This was proved by Denis [**21**] by elementary means, by taking advantage that each function f_s satisfies an inhomogeneous linear Mahler equation of order one. As described in [**60, 61**], this approach can even be pushed forward in order to describe all algebraic relations between the numbers $\zeta_C(s)$. In comparison, it is conjectured that the real numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are all algebraically independent over $\overline{\mathbb{Q}}$, but it remains unknown whether $\zeta(3)$ is transcendental or not, and if $\zeta(5)$ is rational or not.

More recently, Fernandes [**25**] obtained a refinement of Theorem 8.1, which is the analogue of Theorem 6.2. In order to ensure that such a refinement holds true, an additional assumption is now needed: the field extension $\overline{K}(z)(f_1(z), \ldots, f_n(z))$ has to be *regular*. The main difference here is the following. When K is a number field, these extensions are always regular, whereas, when the characteristic of the field K divides q, non-regular q-Mahler extensions do exist. Furthermore, it is shown in [**25**] that the regularity of the field extension $\overline{K}(z)(f_1(z), \ldots, f_n(z))$ is also a necessary condition for the refinement to hold.

9. Final comments

We end this survey with few historical remarks. We have tried, as best as possible, to trace the main steps in the development of this theory, from Mahler's first papers up to now.

9.1. From 1929 to 1969. — Mahler first theorems were quite an achievement as among the very first ones concerning the transcendence and algebraic independence at algebraic points of a whole class of (non-explicit) analytic functions. But the very least one can say is that Mahler did not received much consideration for his original work. His first three papers were totally ignored for more than forty years. Concerning this matter, Mahler wrote for instance in [**44**]: *E. Landau did not show much interest in this result. So I next turned to a closer study of the approximation properties of e and π.* Mahler himself seems to have somewhat underestimate the importance of his own work on this topic all along his life, as filters through [**42, 43, 44**]. The fact that his first paper on the subject appeared in print in 1929, that is the same year as the landmark paper of Siegel [**73**], is maybe not unrelated to the lack of recognition of his method. Indeed, there is no doubt that the theory of the Siegel E-functions got his success because it can be applied to the transcendence of numbers such as e and π, and more generally to the transcendence of values at algebraic points of classical analytic functions such as some hypergeometric series or Bessel functions. In contrast, no classical transcendental constant is known (and even expected) to be the value at some algebraic point of an M-function. This is certainly a major deficiency that this method has. Mahler still noticed at least two interesting instances of such functional equations, but he was not very lucky. The first one is the algebraic modular equation $\Phi_q(J(z), J(z^q)) = 0$, already discussed in Section 5. The second one occurs when studying some theta type functions such as:

$$\theta(z,q) = \sum_{n=0}^{\infty} z^{2n} q^{n^2}$$

which satisfies the functional equation $\theta(zq,q) = (\theta(z,q) - 1)/z^2 q$. Unfortunately, Mahler's method fails to apply in both cases. In the first one the degree of Φ_q is too large (see Theorem 5.2), while in the second case the spectral radius of the underlying matrix transformation is equal to one, so that the latter does not belong to the class $\mathcal{M}$.

This could have mark the end of a brief history, but the theory restarted somewhat accidentally in 1969. After W. Schwarz wrote a paper in which he reproved some results covered by far by Mahler's results, Mahler [**40**] wrote an article in order to inform the mathematical community about his old results and to suggest three main problems of research. It was published in English in the first issue of the *Journal of Number Theory* and was a turning point for Mahler's method.

9.2. From 1969 to 1996. — After Mahler published his paper [**40**], Kubota and independently Loxton and van der Poorten first generalized his old results and popularized the method among number theorists during the second half of the Seventies. They were later joined by other mathematicians including Amou, Becker, Dumas, Flicker, Galochkin, Masser, Miller, Molchanov, Nesterenko, Ku. Nishioka, Randé, Tanaka, Töpfer, Wass. Mahler's method then became an active area of research. The interested reader will find in the book of Ku. Nishioka [**56**] an exhaustive account of Mahler's method during this period, as well as many references. We still mention the useful *Thèse de doctorat* of Dumas [**23**] and Bezivin's paper [**15**] that are not quoted in [**56**]. This period culminated notably with partial or complete solutions to the three problems raised by Mahler in [**40**]:

- Masser [**45**] proved his vanishing theorem (reshaped in this text as Theorem 2.6), which brings a complete solution to the first problem.
- Loxton and van der Poorten [**33**] contributed to the second problem which concerns the generalization of Mahler's method to infinite chains of equations of the form:

  $$f_r(\mathbf{z}) = a_r(\mathbf{z}) f_{r+1}(\Omega \mathbf{z}) + b_r(\mathbf{z}) \qquad (r = 1, 2, 3, \ldots).$$

  In particular, they succeeded in extending Corollary 4.2 to all irrational numbers ω.
- Ku. Nishioka [**51, 52**] contributed to the third problem concerning the algebraic Mahler equation which is discussed in Section 5.

Of course, one should also add to this list Nishioka's theorem (Theorem 6.1) which is one of the jewel of this theory. As it is often the case in transcendence theory, arguments in proofs can be quantified in order to derive transcendence measures, measures of algebraic independence, or to apply to some transcendental points. This was developed by different authors including Nesterenko, Ku. Nishioka, Amou, Becker, and Töpfer. Together with Ku. Nishioka, Loxton and van der Poorten were certainly among the main players in Mahler's method at that time. The latter did a lot to promulgate this topic and also to envisage the main problems that should fall under the scope of this method. Unfortunately, they also were sometimes a little bit too optimistic and published flawed results several times. This makes rather difficult to precisely trace the development of this theory for an outsider, and explains why weaker statements than some announced by these authors were sometimes published much later by others. In this direction, we stress that the recent papers [**4, 6**] provide now rigorous proofs for the results claimed by Loxton and van der Poorten in [**34, 35, 68**].

9.3. Form 1996 up to now. — Of course, after Ku. Nishioka wrote her book, nice applications of Mahler's method were obtained over the years. Let us mention for example the works of Masser [**46**] and Pellarin [**59**] related to Corollary 4.2, the extension by Philippon [**64**] of Nishioka's theorem to transcendental points, and the work of Denis [**20, 21**] concerning Mahler's method in positive characteristic. But Mahler's method really found a great renewed interest in the last few years. On the one hand, the works of Philippon [**62**], and of Faverjon and the author [**4, 5, 6**] provide notable advances concerning linear and algebraic independence of values of M-functions at algebraic points (see also [**13**]). On the other hand, with a complementary point of view, there is an impressive number of recent papers about the algebraic relations between M-functions themselves and the study of the one-variable linear Mahler equation using ideas from difference Galois theory [**1, 12, 16, 18, 22, 57, 58, 71, 70, 72**]. All these recent results give up a glimpse of an exciting period for the future development of Mahler's method.

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