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A new proof of Nishioka's theorem in Mahler's method

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Abstract. In a recent work [3], the authors established new results about general linear Mahler systems in several variables from the perspective of transcendental number theory, such as a multivariate extension of Nishioka's theorem. Working with functions of several variables and with different Mahler transformations leads to a number of complications, including the need to prove a general vanishing theorem and to use tools from ergodic Ramsey theory and Diophantine approximation (e.g., a variant of the p-adic Schmidt subspace theorem). These complications make the proof of the main results proved in [3] rather intricate. In this article, we describe our new approach in the special case of linear Mahler systems in one variable. This leads to a new, elementary, and self-contained proof of Nishioka's theorem, as well as of the lifting theorem more recently obtained by Philippon [23] and the authors [1]. Though the general strategy remains the same as in [3], the proof turns out to be greatly simplified. Beyond its own interest, we hope that reading this article will facilitate the understanding of the proof of the main results obtained in [3].

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1. Introduction

Throughout this paper, we let $q \geq 2$ denote a fixed integer. An M_q*-function* is a power series $f(z) \in \overline{\mathbb{Q}}[\![z]\!]$ satisfying a linear equation of the form

$$p_0(z)f(z) + p_1(z)f(z^q) + \cdots + p_m(z)f(z^{q^m}) = 0\,,$$

where $p_0(z),\ldots,p_m(z) \in \overline{\mathbb{Q}}[z]$ are not all zero. In the study of M_q-functions, it is often more convenient to consider, instead of linear Mahler equations, linear systems of functional equations of the form

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z^q) \\ \vdots \\ f_m(z^q) \end{pmatrix}, \tag{1}$$

where $A(z) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z))$ and $f_1(z),\ldots,f_m(z) \in \overline{\mathbb{Q}}[\![z]\!]$. Then, each power series $f_i(z)$ is an M_q-function. We recall that an M_q-function is meromorphic in the open unit disc of $\mathbb{C}$ (see, for

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instance, [9, Théorème 31]). Furthermore, it admits the unit circle as a natural boundary, unless it is a rational function [24, Théorème 4.3]. A point $\alpha \in \mathbb{C}$ is said to be *regular* with respect to (1) if the matrix $A(\alpha^{q^k})$ is both well-defined and invertible for all integers $k \geq 0$.

In this framework, the main aim of Mahler's method is to transfer results about the absence of algebraic (resp. linear) relations between the functions $f_1(z), \ldots, f_m(z)$ over $\overline{\mathbb{Q}}(z)$ to the absence of algebraic (resp. linear) relations over $\overline{\mathbb{Q}}$ between their values at non-zero algebraic points lying in the open unit disc (assuming, of course, that these values are well-defined). In 1990, Ku. Nishioka [19] proved the following theorem, which is the analog of the Siegel–Shidlovskii theorem in the theory of Siegel E-functions (see [25]). Given a field $\mathbb{K}$, a field extension $\mathbb{L}$ of $\mathbb{K}$, and elements $a_1, \ldots, a_m$ in $\mathbb{L}$, we let $\operatorname{tr.deg}_{\mathbb{K}}(a_1, \ldots, a_m)$ denote the transcendence degree over $\mathbb{K}$ of the field extension $\mathbb{K}(a_1, \ldots, a_m)$.

Theorem 1 (Nishioka's theorem). *Let $f_1(z), \ldots, f_m(z)$ be M_q-functions related by a Mahler system of the form* (1) *and let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, be regular with respect to this system. Then*

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha), \ldots, f_m(\alpha)) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z), \ldots, f_m(z)).$$

Nishioka's theorem is undoubtedly a landmark result in Mahler's method, but it also suffers from some limitations which prevent it from covering important applications (see the discussion in Sections 1 and 2 of [1] and also the results in [2]). For such applications, the following refinement of Nishioka's theorem, which we called *lifting theorem* (or *théorème de permanence* in French), is needed.

Theorem 2 (Lifting theorem). *Let $f_1(z), \ldots, f_m(z)$ be M_q-functions related by a Mahler system of the form* (1) *and let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, be regular with respect to this system. Then for any homogenous polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that*

$$P(f_1(\alpha), \ldots, f_m(\alpha)) = 0,$$

there exists a polynomial $\overline{P} \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_m]$, homogeneous in $X_1, \ldots, X_m$, such that

$$\overline{P}(z, f_1(z), \ldots, f_m(z)) = 0 \text{ and } \overline{P}(\alpha, X_1, \ldots, X_m) = P(X_1, \ldots, X_m).$$

Again, Theorem 2 has an analog in the theory of E-functions: the lifting theorem proved by Beukers [8] using André's theory of arithmetic Gevrey series [5, 6]. A slightly weaker version of Theorem 2 was first proved by Philippon [23]. Theorem 2 was then deduced in [1] from Philippon's lifting theorem. In [1, 23], the lifting theorem is derived from Nishioka's theorem. Thanks to the work of André [7], pursued by Naguy and Szamuely [17], we now have a general approach based on a suitable Galois theory of linear differential and difference equations that allows one to deduce theorems of the type of Theorem 2 from theorems of the type of Theorem 1.

The proof of Nishioka's theorem deeply relies on tools from commutative algebra, related to elimination theory, which were introduced and developed by Nesterenko in the framework of transcendental number theory at the end of the 1970s (see, for instance, [18]). Recently, Fernandes [11] observed that Nishioka's theorem can also be derived from a general algebraic independence criterion due to Philippon [22]. However, Philippon's criterion is also based on the same tools, so that, in the end, both proofs rely on the same argument. The proof of Nishioka's theorem has the advantage that it can be quantified (see, for instance, [19]), leading to algebraic independence measures. Its main deficiency is that it can hardly be generalised to Mahler systems in several variables.

In this note, we use the approach recently introduced by the authors [3] to provide new and more elementary proofs of both Nishioka's theorem and the lifting theorem. This approach takes its roots in the original one initiated by Mahler [16] and developed much later by Kubota [13], Loxton and van der Poorten [15], and Nishioka [20, 21]. The main improvement comes from the introduction of the so-called *relation matrices* whose existence is ensured by Hilbert Nullstellensatz. In contrast with [1, 23], we first prove the lifting theorem and then deduce Nishioka's theorem by using a classical argument, as in Shidlovskii's proof of the Siegel–Shidlovskii theorem (see [10] or [25]). Beyond its elementary aspect, this new approach has the great advantage of being generalisable within the framework of Mahler's method in several variables, as has been done in [3]. We hope that reading first this article will facilitate the understanding of the proof of the main results in [3].

2. Lifting the linear relations

We first prove Theorem 2 in the particular case of linear relations.

Theorem 3. *Let $f_1(z),\dots,f_m(z)$ be M_q-functions related by a Mahler system of the form* (1)*, and let $\alpha \in \overline{\mathbb{Q}}$, $0<|\alpha|<1$, be regular with respect to this system. Let $L \in \overline{\mathbb{Q}}[X_1,\dots,X_m]$ be a linear form such that*

$$L(f_1(\alpha),\dots,f_m(\alpha)) = 0\,.$$

Then, there exists $\overline{L} \in \overline{\mathbb{Q}}[z, X_1,\dots,X_m]$, linear in $X_1,\dots,X_m$, such that

$$\overline{L}(z, f_1(z),\dots,f_m(z)) = 0 \quad \textit{and} \quad \overline{L}(\alpha, X_1,\dots,X_m) = L(X_1,\dots,X_m)\,.$$

The proof of this theorem is divided into three subsections. We first establish the existence and properties of some special matrices which can be associated with a linear Mahler system. We call them the *relation matrices*. Then, we construct an auxiliary function and use it to prove a *key lemma* about the structure of the linear relations between $f_1(z),\dots,f_m(z)$. Finally, we show how this lemma allows us to lift any linear relation over $\overline{\mathbb{Q}}$ between $f_1(\alpha),\dots,f_m(\alpha)$ into a linear relation over $\overline{\mathbb{Q}}(z)$ between $f_1(z),\dots,f_m(z)$. Throughout this section, we keep the notation of Theorem 3.

2.1. *Notation*

Let d be a positive integer and R be a commutative ring. Given an indeterminate x, we let $R[\![x]\!]$ denote the ring of formal power series with coefficients in R. If $R \subset \mathbb{C}$, we let $R\{x\}$ denote the ring of convergent power series with coefficients in R, that is those elements of $R[\![x]\!]$ that are analytic in some neighborhood of the origin. Given a d-tuple of non-negative integers $\boldsymbol{k} = (k_1,\dots,k_d)$, we set $|\boldsymbol{k}| := k_1 + \cdots + k_d$. If $X_1,\dots,X_d$ are indeterminates, we set $\boldsymbol{X}^{\boldsymbol{k}} := X_1^{k_1}\cdots X_d^{k_d}$. The total degree of a polynomial in $R[X_1,\dots,X_d]$ is defined by

$$\deg\left(\sum_{\boldsymbol{k}\in K} a_{\boldsymbol{k}}\boldsymbol{X}^{\boldsymbol{k}}\right) := \max\{|\boldsymbol{k}| : \boldsymbol{k} \in K\,,\, a_{\boldsymbol{k}} \neq 0\}\,.$$

Given an $m \times n$ matrix $M := (m_{i,j})$ with coefficients in R and an $m \times n$ matrix $\boldsymbol{\mu} = (\mu_{i,j})$ with nonnegative integer coefficients, we set

$$M^{\boldsymbol{\mu}} := \prod_{\substack{1\le i\le m\\ 1\le j\le n}} m_{i,j}^{\mu_{i,j}}\,.$$

We use the standard Landau notation $\mathcal{O}$. We also use the notation $\gg$ as follows. Writing that some property holds for all integers $\lambda \gg 1$ means that the corresponding property holds for all λ large enough; writing that some property holds for all integers $\lambda_1 \gg \lambda_2, \lambda_3$ means that the corresponding property holds for all λ_1 that is sufficiently large with respect to λ_2 and λ_3; writing that some property holds for all integers $\lambda_1 \gg \lambda_2 \gg \lambda_3$ means that the corresponding property holds for all λ_1 that is sufficiently large with respect to λ_2, assuming that λ_2 is itself sufficiently large with respect to λ_3.

2.2. *Relation matrices*

To shorten the notation, we set

$$\boldsymbol{f}(z) := (f_1(z), \ldots, f_m(z))^\top .$$

For every integer $k \geq 0$, we set

$$A_k(z) := A(z)A(z^q) \cdots A(z^{q^{k-1}}),$$

so that $A_0(z) = \mathrm{I}_m$, the identity matrix of size m, $A_1(z) = A(z)$, and

$$\boldsymbol{f}(z) = A_k(z)\boldsymbol{f}(z^{q^k}), \qquad \forall\, k \geq 0. \tag{2}$$

Let $\boldsymbol{Y} := (y_{i,j})_{1\leq i,j\leq m}$ denote a matrix of indeterminates. Given a field $\mathbb{K}$ and a non-negative integer δ_1, we let $\mathbb{K}[\boldsymbol{Y}]_{\delta_1}$ denote the set of polynomials of degree at most δ_1 in each indeterminate $y_{i,j}$. Given two non-negative integers δ_1 and δ_2, we let $\mathbb{K}[\boldsymbol{Y}, z]_{\delta_1,\delta_2}$ denote the set of polynomials $P \in \mathbb{K}[\boldsymbol{Y}, z]$ of degree at most δ_1 in every indeterminate $y_{i,j}$ and of degree at most δ_2 in z. The identity theorem and the fact that α is a regular point with respect to (1) ensure that every polynomial $P \in \overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$ is well-defined at the point $(A_k(\alpha), \alpha^{q^k})$ for all $k \gg 1$. Set

$$\mathscr{I} := \left\{P \in \overline{\mathbb{Q}}(z)[\boldsymbol{Y}] \;:\; P(A_k(\alpha), \alpha^{q^k}) = 0,\ \forall\, k \gg 1\right\}.$$

2.2.1. *Estimates for the dimension of certain vector spaces*

Let δ_1 and δ_2 be two non-negative integers. Set $\mathscr{I}(\delta_1) := \mathscr{I} \cap \overline{\mathbb{Q}}(z)[\boldsymbol{Y}]_{\delta_1}$ and $\mathscr{I}(\delta_1,\delta_2) := \mathscr{I} \cap \overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1,\delta_2}$. Note that $\mathscr{I}(\delta_1,\delta_2)$ is a vector subspace of the finite dimensional $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1,\delta_2}$, and let $\mathscr{I}^\perp(\delta_1,\delta_2)$ denote a complement to $\mathscr{I}(\delta_1,\delta_2)$ in $\overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1,\delta_2}$.

Lemma 4. *There exists a positive integer $c_1(\delta_1)$, that does not depend on δ_2, such that*

$$\dim_{\overline{\mathbb{Q}}} \mathscr{I}^\perp(\delta_1,\delta_2) \sim c_1(\delta_1)\delta_2\,, \textit{ as } \delta_2 \textit{ tends to infinity.}$$

Proof. Set $h := (\delta_1+1)^{m^2}$ and let $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_h$ denote an enumeration of the set of all matrices in $\mathcal{M}_m(\mathbb{Z}_{\geq 0})$ whose entries are at most δ_1. Any polynomial $P \in \overline{\mathbb{Q}}(z)[\boldsymbol{Y}]_{\delta_1}$ has a unique decomposition of the form

$$P(\boldsymbol{Y}, z) := \sum_{j=1}^{h} p_j(z)\boldsymbol{Y}^{\boldsymbol{\nu}_j},$$

where $p_j(z) \in \overline{\mathbb{Q}}(z)$, $1 \leq j \leq h$. Since, by definition, $\mathscr{I}(\delta_1)$ does not contain any non-zero elements of $\overline{\mathbb{Q}}$, it is a strict $\overline{\mathbb{Q}}(z)$-subspace of $\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]_{\delta_1}$. Thus, there exist an integer $d \geq 1$ and d vectors of polynomials $(b_{i,1}(z), \ldots, b_{i,h}) \in \overline{\mathbb{Q}}[z]^h$, $1 \leq i \leq d$, which are linearly independent over $\overline{\mathbb{Q}}(z)$ and such that for all $p_1(z), \ldots, p_h(z) \in \overline{\mathbb{Q}}(z)$:

$$\sum_{j=1}^{h} p_j(z)\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \mathscr{I}(\delta_1) \Longleftrightarrow \sum_{j=1}^{h} b_{i,j}(z)p_j(z) = 0 \quad \forall\, i,\ 1 \leq i \leq d. \tag{3}$$

Since these polynomials only depend on δ_1 (and $\mathscr{I}$), there exists $\delta_1' \geq 0$, which only depends on δ_1 (and $\mathscr{I}$), such that

$$b_{i,j}(z) =: \sum_{\kappa=0}^{\delta_1'} b_{i,j,\kappa} z^\kappa, \qquad b_{i,j,\kappa} \in \overline{\mathbb{Q}}.$$

Let us consider $P(\boldsymbol{Y}, z) := \sum_{j=1}^{h} p_j(z)\boldsymbol{Y}^{\boldsymbol{\nu}_j} \in \overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1,\delta_2}$ and set

$$p_j(z) =: \sum_{\lambda\in\mathbb{Z}} p_{j,\lambda} z^\lambda,$$

where the numbers $p_{j,\lambda}$ belong to $\overline{\mathbb{Q}}$ and $p_{j,\lambda}=0$ if $\lambda>\delta_2$ or $\lambda<0$. By (3), P belongs to $\mathscr{I}(\delta_1,\delta_2)$ if and only if

$$\sum_{j=1}^{h}\sum_{\kappa=0}^{\delta_1'} b_{i,j,\kappa}\, p_{j,\gamma-\kappa}=0\,, \qquad \forall\,(\gamma,i)\,,\, 0\le\gamma\le\delta_2+\delta_1'\,,\, 1\le i\le d\,. \tag{4}$$

The number of linearly independent equations in (4) is equal to the dimension of $\mathscr{I}^{\perp}(\delta_1,\delta_2)$. As δ_2 tends to infinity, it is equivalent to the number of linearly independent equations in (4) such that $\delta_1'\le\gamma\le\delta_2$. When γ, $\delta_1'\le\gamma\le\delta_2$, is fixed, the number of linearly independent equations in (4) does not depend on γ. Hence there exists a positive integer $c(\delta_1)$ which does not depend on δ_2 such that

$$\dim_{\overline{\mathbb{Q}}}\mathscr{I}^{\perp}(\delta_1,\delta_2)\sim c(\delta_1)\delta_2\,, \quad \text{as } \delta_2 \text{ tends to infinity.}$$

Note that a more detailed argument can also be found in [4, Section A.1]. □

Lemma 5. *For every pair of non-negative integers (δ_1,δ_2), one has*

$$\dim_{\overline{\mathbb{Q}}}\mathscr{I}^{\perp}(2\delta_1,\delta_2)\le 2^{m^2}\dim_{\overline{\mathbb{Q}}}\mathscr{I}^{\perp}(\delta_1,\delta_2)\,.$$

Proof. Every $P\in\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{2\delta_1,\delta_2}$ can be decomposed as

$$P(\boldsymbol{Y},z)=\sum_{\ell=1}^{2^{m^2}} e_\ell(\boldsymbol{Y})^{\delta_1}P_\ell(\boldsymbol{Y},z)\,, \tag{5}$$

where we let $e_1(\boldsymbol{Y}),\ldots,e_{2^{m^2}}(\boldsymbol{Y})$ denote the 2^{m^2} distinct monomials of degree at most 1 in each $y_{i,j}$, and where the polynomials $P_\ell(\boldsymbol{Y},z)$ all belong to $\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{\delta_1,\delta_2}$. If each polynomial P_ℓ belongs to $\mathscr{I}(\delta_1,\delta_2)$ then $P\in\mathscr{I}(2\delta_1,\delta_2)$. Hence, the decomposition (5) defines a linear map

$$\left(\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{\delta_1,\delta_2}/\mathscr{I}(\delta_1,\delta_2)\right)^{2^{m^2}} \longrightarrow \overline{\mathbb{Q}}[\boldsymbol{Y},z]_{2\delta_1,\delta_2}/\mathscr{I}(2\delta_1,\delta_2)$$

that is surjective. The result follows. □

2.2.2. *Nullstellensatz and relation matrices*

In this section, we show how Hilbert's Nullstellensatz allows us to ensure the existence of a matrix $\boldsymbol{\phi}$, whose entries are all algebraic over $\overline{\mathbb{Q}}(z)$, and which we call a *relation matrix*. Such a matrix encodes the linear relations over $\overline{\mathbb{Q}}(z)$ between the functions $f_1(z),\ldots,f_m(z)$ and is the cornerstone of the proof of Theorem 3.

We first prove the following lemma.

Lemma 6. *The set $\mathscr{I}$ is a radical ideal of $\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$.*

Proof. Checking that $\mathscr{I}$ is an ideal of $\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$ is not difficult. If $P_1,P_2\in\mathscr{I}$, then P_1+P_2 vanishes at $(A_k(\alpha),\alpha^{q^k})$ for all $k\gg1$ and hence $P_1+P_2\in\mathscr{I}$. Now let $P_1\in\mathscr{I}$ and $P_2\in\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$. On the one hand, $P_1(A_k(\alpha),\alpha^{q^k})=0$ for all $k\gg1$ and $P_2(\boldsymbol{Y},z)$ is well-defined at $(A_k(\alpha),\alpha^{q^k})$ for $k\gg1$. We deduce that

$$P_1(A_k(\alpha),\alpha^{q^k})P_2(A_k(\alpha),\alpha^{q^k})=0 \qquad \forall\, k\gg1\,.$$

Hence $P_1P_2\in\mathscr{I}$. Let $P\in\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$ be such that $P^r\in\mathscr{I}$ for some r. If k is a non-negative integer such that $P(A_k(\alpha),\alpha^{q^k})^r=0$, then $P(A_k(\alpha),\alpha^{q^k})=0$. Hence $P\in\mathscr{I}$ and $\mathscr{I}$ is a radical ideal. □

Throughout this article, we let $\mathbb{A}\subset\bigcup_{d\ge1}\overline{\mathbb{Q}}((z^{1/d}))$ denote the algebraic closure of $\overline{\mathbb{Q}}(z)$ in the field of Puiseux series. By the Newton–Puiseux Theorem, $\mathbb{A}$ is algebraically closed.

Lemma 7. *There exists a matrix $\boldsymbol{\phi}(z)\in\mathrm{GL}_m(\mathbb{A})$ such that*

$$P(\boldsymbol{\phi}(z),z)=0\,,$$

for all polynomials $P\in\mathscr{I}$.

Proof. Let us consider the affine algebraic set $\mathcal{V}$ associated with the radical ideal $\mathcal{I}$. That is,

$$\mathcal{V} := \{\boldsymbol{\phi}(z) \in \mathcal{M}_m(\mathbb{A}) \,:\, P(\boldsymbol{\phi}(z), z) = 0 \,,\ \forall\, P \in \mathcal{I}\}\,.$$

According to the weak form of Hilbert's Nullstellensatz (see, for instance, [14, Theorem 1.4, p. 379]), $\mathcal{V}$ is non-empty as soon as $\mathcal{I}$ is a proper ideal of $\overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$. But the definition of $\mathcal{I}$ implies that non-zero constant polynomials do not belong to $\mathcal{I}$. Hence $\mathcal{V}$ is non-empty.

Now, let us assume by contradiction that $\det \boldsymbol{\phi}(z) = 0$ for all $\boldsymbol{\phi}(z)$ in $\mathcal{V}$. By Hilbert's Nullstellensatz (see, for instance, [14, Theorem 1.5, p. 380]), the polynomial $\det \boldsymbol{Y}$ belongs to the radical of the ideal $\mathcal{I}$. Hence $\det \boldsymbol{Y} \in \mathcal{I}$ for $\mathcal{I}$ is radical. Thus, $\det A_k(\alpha) = 0$ for $k \gg 1$. This provides a contradiction since $A_k(\alpha)$ is invertible for all $k \geq 0$. We thus deduce that there exists an invertible matrix $\boldsymbol{\phi}(z)$ in $\mathcal{V}$, as wanted. □

Definition 8. *A matrix $\boldsymbol{\phi}(z) \in \mathrm{GL}_m(\mathbb{A})$ satisfying the property of Lemma 7 is called a* relation matrix.

The next lemma plays a central role in the proof of Theorem 3.

Lemma 9. *Let $\boldsymbol{\phi}(z) \in \mathrm{GL}_m(\mathbb{A})$ be a relation matrix. Then*

$$P\left(\boldsymbol{\phi}(z) A_k(z), z^{q^k}\right) = 0\,,$$

for all $P \in \mathcal{I}$ and all $k \geq 0$.

Proof. Let $P \in \mathcal{I}$, $\boldsymbol{\phi}(z) \in \mathrm{GL}_m(\mathbb{A})$ be a relation matrix, and k be a non-negative integer. Set $Q(\boldsymbol{Y}, z) := P(\boldsymbol{Y} A_k(z), z^{q^k}) \in \overline{\mathbb{Q}}(z)[\boldsymbol{Y}]$. For every $\ell \gg 1$, the polynomial $Q(\boldsymbol{Y}, \alpha^{q^\ell})$ is well-defined and we have

$$Q(A_\ell(\alpha), \alpha^{q^\ell}) = P(A_\ell(\alpha) A_k(\alpha^{q^\ell}), (\alpha^{q^\ell})^{q^k}) = P(A_{k+\ell}(\alpha), \alpha^{q^{k+\ell}}) = 0\,,$$

since $A_\ell(\alpha) A_k(\alpha^{q^\ell}) = A_{k+\ell}(\alpha)$. Hence $Q \in \mathcal{I}$ and

$$P(\boldsymbol{\phi}(z) A_k(z), z^{q^k}) = Q(\boldsymbol{\phi}(z), z) = 0\,,$$

as wanted. □

2.2.3. *Analyticity and relation matrices*

We address now the question of the analyticity of relation matrices.

Lemma 10. *Let $\boldsymbol{\phi}(z) \in \mathrm{GL}_m(\mathbb{A})$ be a relation matrix. Then the three following properties holds for $k \gg 1$.*

(a) *The point α^{q^k} belongs to the disc of convergence of each of the functions $f_1(z), \ldots, f_m(z)$.*
(b) *Each entry of $\boldsymbol{\phi}(z)$ defines an analytic function on some neighborhood of α^{q^k}.*
(c) *The matrix $\boldsymbol{\phi}(\alpha^{q^k})$ is invertible.*

Proof. Since $\lim_{k\to\infty} \alpha^{q^k} = 0$ and $f_1(z), \ldots, f_m(z)$ are analytic on some neighborhood of 0, Property (a) holds for $k \gg 1$. Recall that an algebraic function has only finitely many singularities and finitely many zeros. Hence, for $k \gg 1$, α^{q^k} is neither a singularity of one of the entries of $\boldsymbol{\phi}(z)$ nor a zero of $\det \boldsymbol{\phi}(z)$. We deduce that Properties (b) and (c) hold for $k \gg 1$. □

2.3. *The key Lemma*

Let

$$L(X_1, \ldots, X_m) =: \sum_{j=1}^{m} \tau_j X_i \in \overline{\mathbb{Q}}[X_1, \ldots, X_r]$$

be defined as in Theorem 3. Set $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_m) \in \overline{\mathbb{Q}}^m$ and $\boldsymbol{X} := (X_1, \ldots, X_m)^\top$, so that $L(\boldsymbol{X}) = \boldsymbol{\tau}\boldsymbol{X}$. Given a matrix of indeterminates $\boldsymbol{Y} := (y_{i,j})_{1\le i,j\le m}$, we set

$$F(\boldsymbol{Y}, z) := \sum_{i,j} \tau_i y_{i,j} f_j(z) = \boldsymbol{\tau}\boldsymbol{Y}\boldsymbol{f}(z) \in \overline{\mathbb{Q}}\{z\}[\boldsymbol{Y}],$$

where we recall that $\boldsymbol{f}(z) := (f_1(z), \ldots, f_m(z))^\top$. Note that F is a linear form in $\boldsymbol{Y}$. Evaluating at (I_m, α), where I_m is the identity matrix of size m, we obtain that

$$F(\mathrm{I}_m, \alpha) = \sum_{i=1}^{m} \tau_i f_i(\alpha) = L(\boldsymbol{f}(\alpha)) = 0. \tag{6}$$

Remark 11. We have $F(\boldsymbol{Y}, z) \in \overline{\mathbb{Q}}[\boldsymbol{Y}, \boldsymbol{f}(z)] \subset \overline{\mathbb{Q}}\{z\}[\boldsymbol{Y}]$. Also, $F(\boldsymbol{Y}, z)$ can be seen as an element of $\overline{\mathbb{Q}}[\boldsymbol{Y}][\![z]\!]$, as we will sometimes do in what follows.

2.3.1. *Iterated relations*

For every $k \ge 0$, Equality (2) implies the following equality in $\mathbb{A}[\boldsymbol{Y}]$:

$$\begin{aligned} F(\boldsymbol{Y}, z) &= \boldsymbol{\tau}\boldsymbol{Y}\boldsymbol{f}(z) \\ &= \boldsymbol{\tau}\boldsymbol{Y} A_k(z)\boldsymbol{f}(z^{q^k}) \\ &= F(\boldsymbol{Y} A_k(z), z^{q^k}). \end{aligned} \tag{7}$$

The point α being regular with respect to (1), we deduce from (6) that

$$F(A_k(\alpha), \alpha^{q^k}) = 0, \qquad \forall\, k \ge 0. \tag{8}$$

2.3.2. *The matrices* $\boldsymbol{\Theta}_k(z)$

From now on, we fix a relation matrix $\boldsymbol{\phi}(z)$ and a non-negative integer k_0 such that the properties of Lemma 10 hold for all $k \ge k_0$. Set

$$\xi := \alpha^{q^{k_0}}. \tag{9}$$

Item (a) in Lemma 10 ensures the existence of a positive real number $r_1 < 1$ such that $0 < |\xi| < r_1$ and such that all the power series $f_1(z), \ldots, f_m(z)$ have a radius of convergence larger than r_1. Then, by Item (b) in the same lemma, we can choose $r_2 > 0$ satisfying $0 < |\xi| + r_2 < r_1$ and such that the coefficients of the matrix $\boldsymbol{\phi}(z)$ are analytic on the disc $\mathscr{D}(\xi, r_2)$. For every $k \ge k_0$, we set

$$\boldsymbol{\Theta}_k(z) := A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{\phi}(z) A_{k-k_0}(z) \tag{10}$$

so that we have $\boldsymbol{\Theta}_k(\xi) = A_k(\alpha)$, for every $k \ge k_0$.

Remark 12. By Lemma 10, the coefficients of $\boldsymbol{\Theta}_{k_0}(z)$ are analytic on the disc $\mathscr{D}(\xi, r_2)$. On the other hand, one has

$$\boldsymbol{\Theta}_k(z) = \boldsymbol{\Theta}_{k-1}(z) A(z^{q^{k-1-k_0}}), \qquad \forall\, k > k_0.$$

This implies that, for every $k \ge k_0$, the coefficients of $\boldsymbol{\Theta}_k(z)$ are analytic on some neighborhood of ξ, that is on some disc $\mathscr{D}(\xi, r_k) \subset \mathscr{D}(\xi, r_2)$. In what follows, we will consider the expression $F(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$. Formally, it is a polynomial in $f_1(z^{q^{k-k_0}}), \ldots, f_m(z^{q^{k-k_0}})$ and the entries of $\boldsymbol{\Theta}_k(z)$. Note that it also defines an analytic function on $\mathscr{D}(\xi, r_k) \subset \mathscr{D}(\xi, r_2)$. In addition, $F(\boldsymbol{\Theta}_{k_0}(z), z)$ is analytic on $\mathscr{D}(\xi, r_2)$. Indeed, the functions $f_1(z), \ldots, f_m(z)$ are analytic on $\mathscr{D}(0, r_1) \supset \mathscr{D}(\xi, r_2)$, while our choice of k_0 ensures that the entries of $\boldsymbol{\Theta}_{k_0}(z)$ are analytic on $\mathscr{D}(\xi, r_2)$.

2.3.3. *The key lemma*

The end of the section is devoted to proof of the following result.

Lemma 13. *One has $F(\mathbf{\Theta}_{k_0}(z), z) = 0$.*

In what follows, we argue by contradiction, assuming that

$$F(\mathbf{\Theta}_{k_0}(z), z) \neq 0\,. \tag{11}$$

We divide the proof of Lemma 13 into the four steps (AF), (UB), (NV), and (LB), following the classical proof scheme in transcendental number theory. In Step (AF) we build an *auxiliary function* by considering some sort of Padé approximant of type I for the first powers of $F(\boldsymbol{Y}, z)$. In Step (UB) we compute some *upper bound* for the absolute value of the evaluation of our auxiliary function at $(A_k(\alpha), \alpha^{q^k})$, for large k, by means of analytic estimates. In Step (NV) we prove that our auxiliary function is *non-vanishing* at $(A_k(\alpha), \alpha^{q^k})$ for infinitely many k. In Step (LB), we provide a *lower bound* for the absolute value of the evaluation of our auxiliary function at $(A_k(\alpha), \alpha^{q^k})$, for infinitely many k, by using Liouville's inequality. Finally, we show that the steps (UB) and (LB) lead to a contradiction.

Step (AF). Given a formal power series $E := \sum_{\lambda \geq 0} e_\lambda(\boldsymbol{Y}) z^\lambda \in \overline{\mathbb{Q}}[\boldsymbol{Y}][\![z]\!]$ and an integer $p > 0$, we let

$$E_p := \sum_{\lambda=0}^{p-1} e_\lambda(\boldsymbol{Y}) z^\lambda \in \overline{\mathbb{Q}}[\boldsymbol{Y}, z]$$

denote the truncation of E at order p with respect to z. We recall that $\mathscr{I}^\perp(\delta_1, \delta_2)$ is a complement to $\mathscr{I}(\delta_1, \delta_2)$ in $\overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1, \delta_2}$.

Lemma 14. *Let $\delta_1 \geq 0$ and $\delta_2 \gg \delta_1$ be two integers. Let $p := \left\lfloor \frac{\delta_1 \delta_2}{2^{m^2+2}} \right\rfloor$. Then there exist polynomials $P_i \in \mathscr{I}^\perp(\delta_1, \delta_2)$, $0 \leq i \leq \delta_1$, not all zero, such that the formal power series*

$$E(\boldsymbol{Y}, z) := \sum_{j=0}^{\delta_1} P_j(\boldsymbol{Y}, z) F(\boldsymbol{Y}, z)^j \in \overline{\mathbb{Q}}[\boldsymbol{Y}][\![z]\!]$$

satisfies $E_p(\mathbf{\Theta}_k(z), z^{q^{k-k_0}}) = 0$ for all $k \geq k_0$.

Proof. Set

$$\mathscr{J}(\delta_1, \delta_2) := \{P \in \overline{\mathbb{Q}}[z, \boldsymbol{Y}] \ : \ P(A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{Y}, z) \in \mathscr{I}(\delta_1, \delta_2)\}\,.$$

The $\overline{\mathbb{Q}}$-vector spaces $\mathscr{J}(\delta_1, \delta_2)$ and $\mathscr{I}(\delta_1, \delta_2)$ have same dimension. This follows directly from the fact that the map

$$\begin{aligned} \overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1, \delta_2} &\longrightarrow \overline{\mathbb{Q}}[\boldsymbol{Y}, z]_{\delta_1, \delta_2} \\ P(\boldsymbol{Y}, z) &\longmapsto P(A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{Y}, z) \end{aligned}$$

is an isomorphism, the matrix $A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}$ being invertible. Furthermore, we have

$$P(\mathbf{\Theta}_k(z), z^{q^{k-k_0}}) = 0\,, \qquad \forall\, P \in \mathscr{J}(\delta_1, \delta_2),\ \forall\, k \geq k_0\,. \tag{12}$$

Indeed, if $P \in \mathscr{J}(\delta_1, \delta_2)$, then $P(A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{Y}, z) \in \mathscr{I}(\delta_1, \delta_2)$, and Lemma 9 implies that

$$P(A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{\phi}(z)A_k(z), z^{q^k}) = 0\,, \qquad \forall\, k \geq 0\,.$$

For $k \geq k_0$, replacing k by $k - k_0$ in the previous equality, we obtain that

$$P(A_{k_0}(\alpha)\boldsymbol{\phi}(\alpha^{q^{k_0}})^{-1}\boldsymbol{\phi}(z)A_{k-k_0}(z), z^{q^{k-k_0}}) = 0\,.$$

By (10), we thus have $P(\mathbf{\Theta}_k(z), z^{q^{k-k_0}}) = 0$.

Let p be as in the lemma and let us consider the three $\overline{\mathbb{Q}}$-linear maps:

$$\begin{cases}\left(\mathscr{I}^{\perp}(\delta_1,\delta_2)\right)^{\delta_1+1}\\ (P_0(\boldsymbol{Y},z),\ldots,P_{\delta_1}(\boldsymbol{Y},z))\end{cases}$$
$$\downarrow$$
$$\begin{cases}\overline{\mathbb{Q}}[\boldsymbol{Y}]_{2\delta_1}[\![z]\!]\\ E(\boldsymbol{Y},z):=\sum_{j=0}^{\delta_1}P_j(\boldsymbol{Y},z)F(\boldsymbol{Y},z)^j\end{cases}$$
$$\downarrow$$
$$\begin{cases}\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{2\delta_1,p-1}\\ E_p(\boldsymbol{Y},z)\end{cases}$$
$$\downarrow$$
$$\begin{cases}\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{2\delta_1,p-1}/\mathscr{J}(2\delta_1,p-1)\\ E_p(\boldsymbol{Y},z)\ \mathrm{mod}\ \mathscr{J}(2\delta_1,p-1)\end{cases}$$

Note that these maps are well-defined. By Lemma 4, the dimension of the $\overline{\mathbb{Q}}$-vector space $\mathscr{I}^{\perp}(\delta_1,\delta_2)$ is at least equal to $\frac{c_1(\delta_1)}{2}\delta_2$, assuming that δ_2 is large enough. It follows that

$$\dim_{\overline{\mathbb{Q}}}\left(\left(\mathscr{I}^{\perp}(\delta_1,\delta_2)\right)^{\delta_1+1}\right)\geq\frac{c_1(\delta_1)}{2}(\delta_1+1)\delta_2\,. \tag{13}$$

For every pair of non-negative integers (u,v), set

$$\overline{\mathscr{J}}(u,v):=\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{u,v}/\mathscr{J}(u,v)\,.$$

Since $\mathscr{J}(\delta_1,\delta_2)$ and $\mathscr{I}(\delta_1,\delta_2)$ have same dimension, Lemma 5 implies that

$$\dim_{\overline{\mathbb{Q}}}\overline{\mathscr{J}}(2\delta_1,p-1)\leq 2^{m^2}\dim_{\overline{\mathbb{Q}}}\overline{\mathscr{J}}(\delta_1,p-1)\,.$$

Now, if δ_2 is sufficiently large, Lemma 4 ensures that

$$\dim_{\overline{\mathbb{Q}}}\overline{\mathscr{J}}(\delta_1,p-1)\leq 2c_1(\delta_1)p\,.$$

On the other hand, the choice of p ensures that

$$2^{m^2}\left(2c_1(\delta_1)p\right)<\frac{c_1(\delta_1)}{2}(\delta_1+1)\delta_2$$

and (13) implies that

$$\dim_{\overline{\mathbb{Q}}}\left(\left(\mathscr{I}^{\perp}(\delta_1,\delta_2)\right)^{\delta_1+1}\right)>\dim_{\overline{\mathbb{Q}}}\left(\overline{\mathbb{Q}}[\boldsymbol{Y},z]_{2\delta_1,p-1}/\mathscr{J}(2\delta_1,p-1)\right).$$

Hence the $\overline{\mathbb{Q}}$-linear map defined by

$$(P_0(\boldsymbol{Y},z),\ldots,P_{\delta_1}(\boldsymbol{Y},z))\longmapsto E_p(\boldsymbol{Y},z)\quad \mathrm{mod}\ \mathscr{J}(2\delta_1,p-1)$$

has a non-trivial kernel. We deduce the existence of polynomials $P_0,\ldots,P_{\delta_1}$ in $\mathscr{I}^{\perp}(\delta_1,\delta_2)$, not all zero, such that $E_p\in\mathscr{J}(2\delta_1,p-1)$. By (12), we obtain that $E_p(\boldsymbol{\Theta}_k(z),z^{q^{k-k_0}})=0$ for all $k\geq k_0$. This ends the proof. □

Let $E\in\overline{\mathbb{Q}}[\boldsymbol{Y}][\![z]\!]$ be a formal power series satisfying the properties of Lemma 14 and let v_0 be the smallest index such that the polynomial P_{v_0} is non-zero. Then the formal power series

$$\mathfrak{E}(\boldsymbol{Y},z):=\sum_{j\geq v_0}P_j(\boldsymbol{Y},z)F(\boldsymbol{Y},z)^{j-v_0}\in\overline{\mathbb{Q}}[\boldsymbol{Y}][\![z]\!]$$

is the auxiliary function that we were looking for. Note that we have

$$\mathfrak{E}(\boldsymbol{Y},z)F(\boldsymbol{Y},z)^{v_0}=E(\boldsymbol{Y},z)\,. \tag{14}$$

Warning. The function $\mathfrak{E}(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$ can be thought of as a simultaneous Padé approximant of type I for the first δ_1th powers of $F(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$. However, we have to be careful: $F(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$ it is not necessarily a power series in z. It is a linear combination of $f_1(z^{q^{k-k_0}}),\ldots,f_m(z^{q^{k-k_0}})$ whose coefficients are only known to be algebraic over $\overline{\mathbb{Q}}(z)$. We only know that $F(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$ is analytic in some neighborhood of the point ξ.

Step (UB). The aim of this step is to prove that there exists a real number $c_2 > 0$ such that

$$|\mathfrak{E}(A_k(\alpha), \alpha^{q^k})| \leq e^{-c_2 q^k \delta_1\delta_2}, \qquad \forall\, k \gg \delta_2 \gg \delta_1 . \tag{15}$$

According to Remark 12, the functions $\mathfrak{E}(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$, $F(\boldsymbol{\Theta}_{k_0}(z), z)^{\nu_0}$, and $E(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}})$ are all analytic on the disc $\mathscr{D}(\xi, r_k)$. Hence they respectively have power series expansions of the form

$$\mathfrak{E}(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}) =: \sum_{\lambda=0}^{+\infty} e_{\lambda,k}(z-\xi)^{\lambda}, \qquad e_{\lambda,k} \in \mathbb{C}, \tag{16}$$

$$F(\boldsymbol{\Theta}_{k_0}(z), z)^{\nu_0} =: \sum_{\lambda=0}^{+\infty} a_{\lambda}(z-\xi)^{\lambda}, \qquad a_{\lambda} \in \mathbb{C}, \tag{17}$$

$$E(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}) =: \sum_{\lambda=0}^{+\infty} \epsilon_{\lambda,k}(z-\xi)^{\lambda}, \qquad \epsilon_{\lambda,k} \in \mathbb{C}. \tag{18}$$

We need the following result whose proof is postponed after the end of the argument for proving our main upper bound (15).

Lemma 15. *Let p be defined as in Lemma 14. There exists a real number $\gamma > 0$ that does not depend on the integers δ_1, δ_2, λ, and k, and such that*

$$|\epsilon_{\lambda,k}| \leq e^{-\gamma q^k p}, \qquad \forall\, k \gg \delta_2 \gg \delta_1, \lambda .$$

Using (7), we get that

$$F(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}) = F(\boldsymbol{\Theta}_{k_0}(z), z), \qquad \forall\, k \geq k_0 .$$

By (14), we thus have

$$\mathfrak{E}(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}) F(\boldsymbol{\Theta}_{k_0}(z), z)^{\nu_0} = E(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}), \tag{19}$$

for all $k \geq k_0$ and all $z \in \mathscr{D}(\xi, r_k)$. We use now our assumption that $F(\boldsymbol{\Theta}_{k_0}(z), z)$ is non-zero (see (11)). There thus exists at least one non-zero coefficient a_λ in (17). Let us consider the least integer λ_0 such that $a_{\lambda_0} \neq 0$. Identifying the coefficients of $(z-\xi)^{\lambda_0}$ in the power series expansion of both sides of (19) with the help of (16), (17), and (18), we obtain that

$$e_{0,k} a_{\lambda_0} = \epsilon_{\lambda_0,k}, \ \ \forall\, k \geq k_0 . \tag{20}$$

Since $\boldsymbol{\Theta}_k(\xi) = A_k(\alpha)$ (see (10)) and a_{λ_0} depends only on δ_1 but not on k, we infer from Lemma 15, Equality (20), and the definition of p (see Lemma 14), the existence of a real number $c_2 > 0$ that does not depend on δ_1, δ_2, and k, such that

$$\begin{aligned} \left|\mathfrak{E}(A_k(\alpha), \alpha^{q^k})\right| &= \left|\mathfrak{E}(\boldsymbol{\Theta}_k(\xi), \xi^{q^{k-k_0}})\right| \\ &= |e_{0,k}| \\ &= |\epsilon_{\lambda_0,k}|/|a_{\lambda_0}| \\ &\leq e^{-c_2 q^k \delta_1\delta_2}, \qquad \forall\, k \gg \delta_2 \gg \delta_1 . \end{aligned}$$

This proves the upper bound (15), as wanted.

Now, it remains to prove Lemma 15.

Proof of Lemma 15. Set

$$G(\boldsymbol{Y},z) := E(\boldsymbol{Y},z) - E_p(\boldsymbol{Y},z) \in \overline{\mathbb{Q}}\{z\}[\boldsymbol{Y}],$$

where p is defined as in Lemma 14. By Lemma 14, we have

$$G(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}) = E(\boldsymbol{\Theta}_k(z), z^{q^{k-k_0}}), \qquad \forall\, k \geq k_0. \tag{21}$$

Let $\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_s$ denote an enumeration of all the $m\times m$ matrices with coefficients in the set $\{0,1,\ldots,2\delta_1\}$. There exists a unique decomposition of the form

$$G(\boldsymbol{Y},z) =: \sum_{i=1}^{s}\sum_{\lambda=p}^{\infty} g_{\lambda,i} z^{\lambda}\boldsymbol{Y}^{\boldsymbol{\nu}_i},$$

where $g_{\lambda,i}\in\overline{\mathbb{Q}}$. For every i, $1\leq i\leq s$, we define the formal power series

$$G_i(z) := \sum_{\lambda=p}^{\infty} g_{\lambda,i} z^{\lambda} \in \overline{\mathbb{Q}}[\![z]\!].$$

By definition of $F(\boldsymbol{Y},z)$, these series belong to $\overline{\mathbb{Q}}[z,\boldsymbol{f}(z)]$. In particular, they are analytic on some disc $\mathscr{D}(0,r)$ with $r>r_1$ (where r_1 is defined at the beginning of Section 2.3.2). From the Cauchy–Hadamard inequality, there exists a positive real number $\gamma_1(\delta_1,\delta_2)$ such that

$$|g_{\lambda,i}| \leq \gamma_1(\delta_1,\delta_2) r_1^{-\lambda}, \qquad \forall\,\lambda\geq 0. \tag{22}$$

For every $k\geq k_0$, $G_i(z^{q^{k-k_0}})$ can thus be written as

$$G_i(z^{q^{k-k_0}}) =: \sum_{\lambda=q^{k-k_0}p}^{\infty} g_{\lambda,i,k} z^{\lambda}, \tag{23}$$

with $g_{\lambda,i,k}\in\overline{\mathbb{Q}}$. Furthermore, this power series is absolutely convergent on the disc $\mathscr{D}(0,r_1)$. Since $r_1\leq 1$, we deduce from (22) that

$$|g_{\lambda,i,k}| \leq \gamma_1(\delta_1,\delta_2) r_1^{-\lambda q^{k_0-k}} \leq \gamma_1(\delta_1,\delta_2) r_1^{-\lambda}, \tag{24}$$

for all $\lambda\geq 0$, $i\in\{1,\ldots,s\}$, and $k\geq k_0$. On the other hand, every function $G_i(z^{q^{k-k_0}})$, $1\leq i\leq s$, $k\geq k_0$, is analytic on the disc $\mathscr{D}(\xi,r_k)$. Thus, we can write

$$G_i(z^{q^{k-k_0}}) =: \sum_{\lambda=0}^{\infty} h_{\lambda,i,k}(z-\xi)^{\lambda}, \tag{25}$$

where $h_{\lambda,i,k}\in\mathbb{C}$. Since by assumption $\mathscr{D}(\xi,r_k)\subset\mathscr{D}(0,r_1)$, the two power series expansions (23) and (25) match on $\mathscr{D}(\xi,r_k)$. Using the equality

$$z^{\gamma} = ((z-\xi)+\xi)^{\gamma} = \sum_{\lambda=0}^{\gamma}\binom{\gamma}{\lambda}\xi^{\gamma-\lambda}(z-\xi)^{\lambda} \tag{26}$$

and identifying, for every $\lambda\geq 0$, the coefficients of $(z-\xi)^{\lambda}$ in (23) and (25), we deduce that

$$h_{\lambda,i,k} = \sum_{\substack{\gamma\geq q^{k-k_0}p\\ \gamma\geq\lambda}}\binom{\gamma}{\lambda} g_{\gamma,i,k}\xi^{\gamma-\lambda}. \tag{27}$$

For $\gamma\geq\lambda$, one has

$$\binom{\gamma}{\lambda} = \frac{\gamma!}{(\gamma-\lambda)!\lambda!} \leq \gamma^{\lambda}. \tag{28}$$

Given $\lambda\geq 0$, we have that $\lambda<q^{k-k_0}p$ as soon as k is large enough, and since $|\xi|<r_1$, we infer from (24), (27) and (28) the existence of a real number $\gamma_2>0$ that does not depend on δ_1, δ_2, λ, and k, such that

$$|h_{\lambda,i,k}| \leq e^{-\gamma_2 q^k p}, \qquad \forall\, k\gg\delta_1,\delta_2,\lambda. \tag{29}$$

Now, we proceed to bound the absolute value of the coefficients of the power series expansion in $z-\xi$ of $\boldsymbol{\Theta}_k(z)^{\boldsymbol{\nu}_i}$, $1\le i\le s$. Given a power series $Q(z)\in\overline{\mathbb{Q}}\{z\}$ and $k\ge 0$, we write

$$Q(z^{q^{k-k_0}}) =: \sum_{\lambda=0}^{\infty} q_{\lambda,k}(z-\xi)^{\lambda}.$$

For all k large enough, $\xi^{q^{k-k_0}}$ belongs to the domain of analyticity of $Q(z)$. Using again (26) and (28) we obtain that, for every $\lambda\ge 0$, $|q_{\lambda,k}|=\mathcal{O}(1)$ as k tends to infinity, where the underlying constant in the $\mathcal{O}$ notation depends both on $Q(z)$ and λ. Fix some $\lambda\ge 0$. Let $v\ge 0$ be an integer such that the entries of $z^{v}A(z)$ have no poles at 0. The entries of $z^{v}A(z)$ are convergent power series at 0, and the points $\xi^{q^{k-k_0}}$ belong to their domain of analyticity for k large enough. Then, the coefficients of $(z-\xi)^{\lambda}$ in the power series expansion in $z-\xi$ of each of the entries of $z^{vq^{k-k_0}}A(z^{q^{k-k_0}})$ belong to $\mathcal{O}(1)$ as k tends to infinity. Using (26), we write

$$\begin{aligned} z^{-vq^{k-k_0}} &= \xi^{-vq^{k-k_0}}\left(1+\sum_{\lambda=1}^{vq^{k-k_0}}\binom{vq^{k-k_0}}{\lambda}\xi^{-\lambda}(z-\xi)^{\lambda}\right)^{-1}\\ &= \xi^{-vq^{k-k_0}}\\ &\qquad+\sum_{\lambda=1}^{\infty}\left(\xi^{-vq^{k-k_0}}\sum_{t=1}^{\lambda}\sum_{\lambda_1+\cdots+\lambda_t=\lambda}\prod_{i=1}^{t}\binom{vq^{k-k_0}}{\lambda_i}\xi^{-\lambda_i}\right)(z-\xi)^{\lambda}\\ &=: \sum_{\lambda=0}^{\infty} r_{\lambda,k}(z-\xi)^{\lambda}. \end{aligned}$$

Using (28), we deduce the existence of a real number $\gamma_3>0$ which does not depend on k and such that $|r_{\lambda,k}|=\mathcal{O}(e^{\gamma_3 q^k})$ as k tends to infinity. It follows that the absolute value of the coefficient of $(z-\xi)^{\lambda}$, in the power series expansion in $z-\xi$ of each of the entries of $A(z^{q^{k-k_0}})$, belongs to $\mathcal{O}(e^{\gamma_3 q^k})$ as k tends to infinity, where the underlying constant in the $\mathcal{O}$ notation depends on λ but not on δ_1, δ_2, and k.

By Remark 12, the monomial $\boldsymbol{\Theta}_k(z)^{\boldsymbol{\nu}_i}$ is analytic on $\mathcal{D}(\xi,r_k)$ for every i, $1\le i\le s$, and every $k\ge k_0$. Thus, we can write

$$\boldsymbol{\Theta}_k(z)^{\boldsymbol{\nu}_i} =: \sum_{\lambda=0}^{+\infty}\theta_{\lambda,i,k}(z-\xi)^{\lambda}, \tag{30}$$

where $\theta_{\lambda,i,k}\in\mathbb{C}$. Using the recurrence relation

$$\boldsymbol{\Theta}_{k+1}(z)=\boldsymbol{\Theta}_k(z)A(z^{q^{k-k_0}}),$$

we obtain the existence of a real number $\gamma_4(\lambda)>0$ that does not depend on i, δ_1, δ_2, and k, such that the absolute value of the coefficient of $(z-\xi)^{\lambda}$ in each of the entries of $\boldsymbol{\Theta}_k(z)$ is at most $e^{\gamma_4(\lambda)q^k}$. Since $|\boldsymbol{\nu}_i|\le 2m^2\delta_1$ for each i, there exists a real number $\gamma_5(\lambda)>0$ that does not depend on i, δ_1, δ_2, and k, such that

$$|\theta_{\lambda,i,k}|<e^{\gamma_5(\lambda)\delta_1 q^k}, \qquad \forall\, i,\ 1\le i\le s,\ \forall\, k\ge k_0. \tag{31}$$

From (18), (21), (25), and (30), we deduce that

$$\sum_{i=1}^{s}\left(\sum_{\lambda=0}^{+\infty}h_{\lambda,i,k}(z-\xi)^{\lambda}\right)\left(\sum_{\lambda=0}^{+\infty}\theta_{\lambda,i,k}(z-\xi)^{\lambda}\right)=\sum_{\lambda=0}^{+\infty}\epsilon_{\lambda,k}(z-\xi)^{\lambda}. \tag{32}$$

Finally, identifying the coefficents of $(z-\xi)^{\lambda}$ in both sides of (32), we have

$$\epsilon_{\lambda,k}=\sum_{i=1}^{s}\sum_{\gamma=0}^{\lambda}h_{\gamma,i,k}\theta_{\lambda-\gamma,i,k}.$$

Note that $p \gg \delta_1$ when $\delta_2 \gg \delta_1$. Inequalities (29) and (31) imply the existence of a real number $\gamma_6 > 0$ that does not depend on δ_1, δ_2, λ, and k, and such that

$$|\epsilon_{\lambda,k}| \leq e^{-\gamma_6 q^k p}, \qquad \forall\, k \gg \delta_2 \gg \delta_1, \lambda.$$

Setting $\gamma := \gamma_6$, this ends the proof. □

Step (NV). Let us first recall that by (8) we have

$$F(A_k(\alpha), \alpha^{q^k}) = 0, \qquad \forall\, k \geq 0.$$

By construction of our auxiliary function, we deduce that

$$\mathfrak{E}(A_k(\alpha), \alpha^{q^k}) = P_{v_0}(A_k(\alpha), \alpha^{q^k}).$$

Furthermore, since this construction ensures that $P_{v_0} \notin \mathscr{I}$, there exists an infinite set of positive integers $\mathscr{E}$ such that

$$P_{v_0}(A_k(\alpha), \alpha^{q^k}) \neq 0, \qquad \forall\, k \in \mathscr{E}.$$

Without any loss of generality, we assume that $k \geq k_0$ for all $k \in \mathscr{E}$.

Step (LB). Given an algebraic number β, we let $h(\beta)$ denote the absolute logarithmic Weil height of β (see [26, Chapter 3] for a definition). In order to prove our lower bound, we only need the following basic properties of the Weil height. The use of the logarithmic Weil height simplifies some computations but any other standard notion of height would also do the job. Given two algebraic numbers β and γ, one has (see [26, Property 3.3]):

$$\begin{aligned} h(\beta+\gamma) &\leq h(\beta) + h(\gamma) + \log 2 \\ h(\beta\gamma) &\leq h(\beta) + h(\gamma) \\ h(\beta^n) &= |n| h(\beta), \quad \beta \neq 0,\ n \in \mathbb{Z}. \end{aligned} \tag{33}$$

Let $P := \sum_{\boldsymbol{k}\in K} a_{\boldsymbol{k}} \boldsymbol{X}^{\boldsymbol{k}} \in \overline{\mathbb{Q}}[X_1,\dots,X_n]$, and $\beta_1,\dots,\beta_n \in \overline{\mathbb{Q}}$, we deduce from [26, Lemma 3.7] that

$$h(P(\beta_1,\dots,\beta_n)) \leq \sum_{i=1}^{n} \log(1+\deg_{X_i}(P)) + \sum_{i=1}^{n} (\deg_{X_i} P) h(\beta_i) + \sum_{\boldsymbol{k}\in K} h(a_{\boldsymbol{k}}). \tag{34}$$

Given a number field $\mathbf{k}$, we have the fundamental *Liouville inequality* (see [26, p. 82]):

$$\log|\beta| \geq -[\mathbf{k}:\mathbb{Q}]\, h(\beta), \qquad \forall\, \beta \neq 0 \in \mathbf{k}. \tag{35}$$

We are going to use (35) to find a lower bound for $|\mathfrak{E}(A_k(\alpha), \alpha^{q^k})|$. A simple computation by induction on k shows that the height of each entry of $A_k(\alpha)$ is at most γq^k for some $\gamma > 0$ that does not depend on k (see [4, Section A.2] for more details). The polynomial $P_{v_0}(\boldsymbol{Y}, z)$ has degree at most δ_1 in each indeterminate $y_{i,j}$ and degree at most δ_2 in z. Furthermore, its coefficients are algebraic numbers which only depend on the parameters δ_1 and δ_2. Using (33) and (34), we obtain that the height of the algebraic number $P_{v_0}(A_k(\alpha), \alpha^{q^k})$ is at most $cq^k\delta_2$ for some constant c that does not depend on δ_1, δ_2, and k, assuming that $k \gg \delta_2 \geq \delta_1$. Since these algebraic numbers belong to a fixed number field, Liouville's inequality ensures the existence of $c_3 > 0$ that does not depend on δ_1, δ_2, and k, and such that

$$|\mathfrak{E}(A_k(\alpha), \alpha^{q^k})| = |P_{v_0}(A_k(\alpha), \alpha^{q^k})| \geq e^{-c_3 q^k \delta_2}, \qquad \forall\, k \in \mathscr{E},\ k \gg \delta_2 \geq \delta_1. \tag{36}$$

We are now ready to end the proof of our key lemma.

Proof of Lemma 13. By Inequalities (15) and (36), we obtain that

$$e^{-c_3 q^k \delta_2} \leq |\mathfrak{E}(A_k(\alpha), \alpha^{q^k})| \leq e^{-c_2 q^k \delta_1 \delta_2}, \qquad \forall\, k \in \mathscr{E},\ k \gg \delta_2 \gg \delta_1.$$

We deduce that

$$c_3 \geq c_2 \delta_1.$$

Since c_2 and c_3 are positive numbers which do not depend on δ_1, this provides a contradiction, as soon as δ_1 is large enough. □

2.4. *End of the proof of Theorem 3*

The entries of $\boldsymbol{\phi}(z)$ being algebraic over $\overline{\mathbb{Q}}(z)$, they generate a finite extension of $\overline{\mathbb{Q}}(z)$. Let $\mathbf{k} \subset \mathbb{A}$ denote this extension and let $\gamma \geq 1$ be the degree of $\mathbf{k}$. We recall that $\mathbb{A}$ is the algebraic closure of $\overline{\mathbb{Q}}(z)$ in the field of Puiseux series. Choosing a primitive element $\varphi(z)$ in $\mathbf{k}$, we obtain a decomposition of the form

$$\boldsymbol{\phi}(z) =: \sum_{j=0}^{\gamma-1} \boldsymbol{\phi}_j(z)\varphi(z)^j , \tag{37}$$

where the matrices $\boldsymbol{\phi}_j(z)$, $0 \leq j \leq \gamma-1$, have coefficients in $\overline{\mathbb{Q}}(z)$. Let $d(z) \in \overline{\mathbb{Q}}[z]$ denote a common denominator of the entries of the matrices $\boldsymbol{\phi}_j(z)$. Without any loss of generality, we can assume that in (9) the integer k_0 has been chosen large enough so that $\varphi(z)$ is analytic at $\xi = \alpha^{q^{k_0}}$ and $d(\alpha^{q^{k_0}}) \neq 0$. Let $q(z)$ denote the least common multiple of the denominators of the entries of the matrix $A_{k_0}^{-1}(z)$. Since α is assumed to be regular with respect to the Mahler system (1), we have that $q(\alpha) \neq 0$.

By Lemma 13, we know that $F(\boldsymbol{\Theta}_{k_0}(z), z) = 0$, and substituting $z^{q^{k_0}}$ for z, we obtain that $F(\boldsymbol{\Theta}_{k_0}(z^{q^{k_0}}), z^{q^{k_0}}) = 0$. The function $F(\boldsymbol{Y}, z)$ being linear in $\boldsymbol{Y}$, we deduce that

$$F\left(\frac{d(z^{q^{k_0}})q(z)}{d(\alpha^{q^{k_0}})q(\alpha)}\boldsymbol{\Theta}_{k_0}(z^{q^{k_0}}), z^{q^{k_0}}\right) = 0 .$$

Writing $\boldsymbol{\Theta}_{k_0}(z^{q^{k_0}}) = \boldsymbol{\Theta}_{k_0}(z^{q^{k_0}})A_{k_0}(z)^{-1}A_{k_0}(z)$ and using (7), we get that

$$F\left(\frac{d(z^{q^{k_0}})q(z)}{d(\alpha^{q^{k_0}})q(\alpha)}\boldsymbol{\Theta}_{k_0}(z^{q^{k_0}})A_{k_0}(z)^{-1}, z\right) = 0 . \tag{38}$$

Now, let us consider the linear form in $X_1, \ldots, X_n$ defined by:

$$Q(z, \boldsymbol{X}) := \boldsymbol{\tau}\left(\frac{d(z^{q^{k_0}})q(z)}{d(\alpha^{q^{k_0}})q(\alpha)}\boldsymbol{\Theta}_{k_0}(z^{q^{k_0}})A_{k_0}(z)^{-1}\right)\boldsymbol{X} .$$

Thus, the coefficient of each X_i in $Q(z, \boldsymbol{X})$ belongs to $\overline{\mathbb{Q}}[z, \varphi(z^{q^{k_0}})]$. Since $\varphi(z^{q^{k_0}})$ is analytic at α, the coefficients of $Q(z, \boldsymbol{X})$ are analytic at α. Moreover, since $\boldsymbol{\Theta}_{k_0}(\alpha^{q^{k_0}}) = \boldsymbol{\Theta}_{k_0}(\xi) = A_{k_0}(\alpha)$, we deduce that

$$Q(\alpha, \boldsymbol{X}) = \boldsymbol{\tau}\boldsymbol{X} = L(\boldsymbol{X}) .$$

Finally, it follows from (38) that

$$Q(z, \boldsymbol{f}(z)) = 0 .$$

There is only one point left to address: we have lifted the linear relation between $f_1(\alpha), \ldots, f_m(\alpha)$ into a linear relation between $f_1(z), \ldots, f_m(z)$, but this relation is over $\overline{\mathbb{Q}}[z, \varphi(z^{q^{k_0}})]$. Since the field $\overline{\mathbb{Q}}(z, \boldsymbol{f}(z))$ is a regular extension of $\overline{\mathbb{Q}}(z)$ (see [1, Lemme 3.2]), we have that $\overline{\mathbb{Q}}(z)(\boldsymbol{f}(z))$ and $\mathbb{A}$ are linearly disjoint over $\overline{\mathbb{Q}}(z)$ (see [14, Chapter VIII]). Let δ denote the degree of $\varphi(z^{q^{k_0}})$ over $\overline{\mathbb{Q}}(z)$, so that the functions $\varphi(z^{q^{k_0}})^j$, $0 \leq j \leq \delta-1$, are linearly independent over $\overline{\mathbb{Q}}(z)$. Since $\overline{\mathbb{Q}}(z)(\boldsymbol{f}(z))$ and $\mathbb{A}$ are linearly disjoint over $\overline{\mathbb{Q}}(z)$, these functions remain linearly independent over $\overline{\mathbb{Q}}(z)(\boldsymbol{f}(z))$. Thus, splitting the linear form Q as

$$Q =: \sum_{j=0}^{\delta-1} Q_j(z, \boldsymbol{X})\varphi(z^{q^{k_0}})^j ,$$

where $Q_j(z,\boldsymbol{X}) \in \overline{\mathbb{Q}}[z,\boldsymbol{X}]$ are linear forms, we deduce that

$$Q_j(z,\boldsymbol{f}(z)) = 0\,, \qquad \forall\, j,\, 0 \le j \le \delta-1\,.$$

Finally, setting

$$\overline{L}(z,\boldsymbol{X}) := \sum_{j=0}^{\delta-1} Q_j(z,\boldsymbol{X})\varphi(\alpha^{q^{k_0}})^j \in \overline{\mathbb{Q}}[z,\boldsymbol{X}]\,,$$

we obtain that $\overline{L}(z,\boldsymbol{f}(z)) = 0$ and $\overline{L}(\alpha,\boldsymbol{X}) = L(\boldsymbol{X})$, as wanted. This ends the proof of Theorem 3. □

3. From linear to algebraic relations

In this section, we deduce Theorem 2 from Theorem 3. The key observation is that, given M_q-functions $f_1(z),\dots,f_m(z)$ related by a Mahler system, the monomials of a given degree in $f_1(z),\dots,f_m(z)$ are also related by a Mahler system with no additional singularities.

Let us first recall some notation. Let $A = (a_{i,j})$ and B be matrices with entries in a given commutative ring, with dimension, respectively, (m,n) and (p,q). The Kronecker product of A and B is the matrix $A \otimes B$, of size (mp,nq) with block decomposition

$$A \otimes B := \begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{pmatrix}.$$

If $d \ge 1$ is an integer, we also set

$$A^{\otimes d} := \underbrace{A \otimes \cdots \otimes A}_{d \text{ times}}.$$

We will use only basic facts about the Kronecker product that can be found in [12] and are also reproved in [4, Section A.3].

Proof of Theorem 2. Let d denote the total degree of P and $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_t$ be an enumeration of the set $\{\boldsymbol{\lambda} \in (\mathbb{Z}_{\ge 0})^m : |\boldsymbol{\lambda}| = d\}$. Then, we have

$$P =: \sum_{j=1}^{t} p_j \boldsymbol{X}^{\boldsymbol{\lambda}_j}\,,$$

where $p_j \in \overline{\mathbb{Q}}$ and $\boldsymbol{X} := (X_1,\dots,X_m)$. Set $\boldsymbol{f}(z) := (f_1(z),\dots,f_m(z))^\top$. The entries of the vector $\boldsymbol{f}(z)^{\otimes d}$ are precisely the monomials of degree d in $f_1(z),\dots,f_m(z)$, with some of them appearing several times (for example, the product $f_1(z)f_2(z)$ appears twice in $\boldsymbol{f}(z)^{\otimes 2}$). Using [12, Lemma 4.2.10] and a straightforward induction on d, we obtain that

$$\boldsymbol{f}(z)^{\otimes d} = A(z)^{\otimes d}\boldsymbol{f}(z^q)^{\otimes d}\,. \tag{39}$$

Since α is a regular point with respect to the system (1) the matrix $A(z)$ is well-defined and invertible at α^{q^k} for all integers $k \ge 0$. The entries of the matrix $A(z)^{\otimes d}$ being products of the entries of $A(z)$, the matrix $A(z)^{\otimes d}$ is well-defined at α^{q^k} for all integers $k \ge 0$. Furthermore, since $\det A(\alpha^{q^k}) \ne 0$ we have $\det A(\alpha^{q^k})^{\otimes d} \ne 0$ (see [12, Corollary 4.2.11]), for all integers $k \ge 0$. Hence α is a regular point with respect to the system (39).

For each j, $1 \le j \le t$, let $\mathscr{I}_j \subset \{1,\dots,m^d\}$ denote the set of integers i for which the ith entry of $\boldsymbol{X}^{\otimes d}$ is $\boldsymbol{X}^{\boldsymbol{\lambda}_j}$. For each j, we pick an integer i_j in $\mathscr{I}_j$. Let $Y_1,\dots,Y_{m^d}$ be a family of indeterminates and let us consider the linear form L defined by

$$L(Y_1,\dots,Y_{m^d}) := \sum_{j=1}^{t} p_j Y_{i_j}\,.$$

We also let $g_1,\dots,g_{m^d}$ denote the entries of $\boldsymbol{f}(z)^{\otimes d}$. By construction $g_i(z)=\boldsymbol{f}(z)^{\boldsymbol{\lambda}_j}$ when $i\in\mathscr{I}_j$. Thus,

$$L(g_1(\alpha),\dots,g_{m^d}(\alpha))=\sum_{j=1}^{t}p_j g_{i_j}(\alpha)=P(f_1(\alpha),\dots,f_m(\alpha))=0\,.$$

By Theorem 3, there exists $\bar{L}\in\overline{\mathbb{Q}}[z,Y_1,\dots,Y_{m^d}]$ linear in $Y_1,\dots,Y_{m^d}$, such that

$$\bar{L}(z,g_1(z),\dots,g_{m^d}(z))=0 \quad \text{and} \quad \bar{L}(\alpha,Y_1,\dots,Y_{m^d})=L(Y_1,\dots,Y_{m^d})\,.$$

Write $\bar{L}=:\sum_{i=1}^{m^d}l_i(z)Y_i$, where $l_0(z),\dots,l_{m^d}(z)\in\overline{\mathbb{Q}}[z]$. We deduce that

$$l_i(\alpha)=\begin{cases}p_j & \text{if } i=i_j \text{ for some } j,\ 1\le j\le t\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Now, set

$$\bar{P}(z,X_1,\dots,X_m):=\sum_{j=1}^{t}\left(\sum_{i\in\mathscr{I}_j}l_i(z)\right)\boldsymbol{X}^{\boldsymbol{\lambda}_j}\,.$$

On the one hand, we have

$$\begin{aligned}\bar{P}(z,f_1(z),\dots,f_m(z))&=\sum_{j=1}^{t}\left(\sum_{i\in\mathscr{I}_j}l_i(z)\right)\boldsymbol{f}(z)^{\boldsymbol{\lambda}_j}\\&=\sum_{i=1}^{m^d}l_i(z)g_i(z)\\&=\bar{L}(z,g_1(z),\dots,g_{m^d}(z))=0\,,\end{aligned}$$

while, on the other hand, we have

$$\bar{P}(\alpha,\boldsymbol{X})=\sum_{j=1}^{t}\left(\sum_{i\in\mathscr{I}_j}l_i(\alpha)\right)\boldsymbol{X}^{\boldsymbol{\lambda}_j}=\sum_{j=1}^{t}p_j\boldsymbol{X}^{\boldsymbol{\lambda}_j}=P(\boldsymbol{X})\,.$$

This ends the proof. □

4. Deducing Nishioka's theorem from the lifting theorem

In this section, we show how to deduce Nishioka's theorem from the lifting theorem.

Proof of Theorem 1. We first note that the inequality

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha),\dots,f_m(\alpha))\le\operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\dots,f_m(z))$$

always holds. Hence we only have to prove that

$$t_\alpha:=\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(f_1(\alpha),\dots,f_m(\alpha))\ge\operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\dots,f_m(z))=:t_z\,. \tag{40}$$

Let $d\ge 0$ be an integer. We let $\varphi_\alpha(d)$ denote the dimension of the $\overline{\mathbb{Q}}$-vector space spanned by the monomials of degree at most d in $f_1(\alpha),\dots,f_m(\alpha)$. We also let $\varphi_z(d)$ denote the dimension of the $\overline{\mathbb{Q}}(z)$-vector space spanned by the monomials of degree at most d in $f_1(z),\dots,f_m(z)$. Note that the functions $1,f_1(z),\dots,f_m(z)$ are related by the Mahler system of size $m+1$:

$$\begin{pmatrix}1\\ f_1(z)\\ \vdots\\ f_m(z)\end{pmatrix}=\left(\begin{array}{c|c}1 & \\ \hline & A(z)\end{array}\right)\begin{pmatrix}1\\ f_1(z^q)\\ \vdots\\ f_m(z^q)\end{pmatrix}\,. \tag{41}$$

Furthermore, the point α remains regular with respect to this new system. Applying Theorem 2 to (41), we obtain that

$$\varphi_\alpha(d)\ge\varphi_z(d),\quad \forall\, d\ge 0\,. \tag{42}$$

By a result of Hilbert, $\varphi_\alpha(d)$ and $\varphi_z(d)$ are polynomials in d of degree respectively equal to t_α and t_z when $d \gg 1$ (see, for instance, the discussion around the Hilbert–Serre theorem in [27, p. 232]). Thus, there exist two positive real numbers β and γ such that

$$\varphi_\alpha(d) \leq \beta d^{t_\alpha} \text{ and } \varphi_z(d) \geq \gamma d^{t_z}, \qquad \forall\, d \gg 1.$$

Using (42), we deduce (40) as wanted. □

Remark 16. In the proof of Theorem 1, we do not need the full strength of Hilbert's result. Suitable estimates for $\varphi_\alpha(d)$ and $\varphi_z(d)$ can be easily achieved by elementary means (see [4, Section A.4]).

Remark 17. At the end of our proof of Theorem 2, we used the fact that the field extension $\overline{\mathbb{Q}}(z, f_1(z),\dots,f_m(z))$ is a regular extension of $\overline{\mathbb{Q}}(z)$. We stress that this argument is not needed to deduce Nishioka's theorem. Indeed, without using it, we still obtain that every $\overline{\mathbb{Q}}$-linear relation between $f_1(\alpha),\dots,f_m(\alpha)$ can be lifted into a linear relation over the algebraic closure $\mathbb{A}$ of $\overline{\mathbb{Q}}(z)$ between $f_1(z),\dots,f_m(z)$. Then we could reproduce the previous argument, just replacing $\overline{\mathbb{Q}}(z)$ by $\mathbb{A}$. We would derive the main result since

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(f_1(z),\dots,f_m(z)) = \operatorname{tr.deg}_{\mathbb{A}}(f_1(z),\dots,f_m(z)),$$

$\mathbb{A}$ being by definition algebraic over $\overline{\mathbb{Q}}(z)$.

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